ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-400 *Proposed by Arne Fransen, Stockholm, Sweden*

For natural numbers h, k, with k odd, and an irrational a in the Lucasian sequence $V_{kh} = a^{kh} + a^{-kh}$, define $y_k \equiv V_{kh}$. Put

$$y_k = \sum_{r=0}^{n} c_r^{(2n+1)} y_1^{(2r+1)}, \text{ with } k = 2n + 1.$$

Prove that the coefficients are given by

$$c_r^{(2n+1)} \begin{cases} \equiv 1 \quad \text{for } r = n, \\ = (-1)^{n-r}(2n+1)\displaystyle\sum_{j=1}^{J} \frac{1}{2j-1}\binom{n-j}{2(j-1)}\binom{n-1-3(j-1)}{r-(j-1)} \quad \text{for } 0 \leqslant r < n, \end{cases}$$

where $J = \min\left(\left[\frac{n+2}{3}\right], \left[\frac{n+1-r}{2}\right], r+1\right)$.

Also, is there a simpler expression for $c_r^{(2n+1)}$?

H-401 *Proposed by Albert A. Mullin, Huntsville, AL*

It is well known that if $n \neq 4$ and the Fibonacci number F_n is prime then n is prime.

(1) Prove or disprove the complementary result: If $n \neq 8$ and the Fibonacci number F_n is the product of two *distinct* primes then n is either prime or the product of two primes, in which case at least one prime factor of F_n is Fibonacci.

(2) Define the recursions $u_{n+1} = F_{u_n}$, $u_1 = F_m$, $m \geqslant 6$. Prove or disprove that each sequence $\{u_n\}$ represents only finitely many primes and finitely many products of two distinct primes.

H-402 *Proposed by Piero Filipponi, Rome, Italy*

A MATRIX GAME (fom the Italian TV serial *Pentathlon*)

Each element of a square matrix M of order 3 is entered with a symbol chosen randomly (with probability 1/2) between two possible symbols (namely x and y). If M contains at least a row (or a column) entirely formed by x's or by y's, then one wins the game.

Generalize to a matrix of order n and find the win probability.

Remark: By inspection, it is easily seen that

$$P_1 = 1,\ P_2 = 7/8, \text{ and } P_3 = 205/256.$$

A computer experiment gave the following results:

$$\begin{array}{ll} P_3 \doteq .801 & P_7 \doteq .200 \\ P_4 \doteq .637 & P_8 \doteq .111 \\ P_5 \doteq .483 & P_9 \doteq .066 \\ P_6 \doteq .325 & P_{10} \doteq .035 \end{array}$$

The conjecture $\lim\limits_{n\to\infty} P_n = 0$ immediately follows.

SOLUTIONS

Late Acknowledgment: *C. Georghiou solved* H-371.

Somewhat Dependable

H-377 *Proposed by Lawrence Somer, Washington, D.C.*
(Vol. 22, no. 4, November 1984)

Let $\{w_n\}_{n=0}^{\infty}$ be a k^{th}-order linear integral recurrence satisfying the recursion relation

$$w_{n+k} = a_1 w_{n+k-1} + a_2 w_{n+k-2} + \cdots + a_k w_n.$$

Let t be a fixed positive integer and d a fixed nonnegative integer. Show that the sequence $\{s_n\} = \{w_{tn+d}\}_{n=0}^{\infty}$ also satisfies a k^{th}-order linear integral recursion relation

$$s_{n+k} = a_1^{(t)} s_{n+k-1} + a_2^{(t)} s_{n+k-2} + \cdots + a_k^{(t)} s_n.$$

Show further that the coefficients $a_1^{(t)}, a_2^{(t)}, \ldots, a_k^{(t)}$ depend on t but not on d, and that $a_k^{(t)}$ can be chosen so that

$$a_k^{(t)} = (-1)^{(k+1)(t+1)} a_k^t.$$

Solution by the proposer

Let

$$f(x) = x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_{k-1} x - a_k \tag{1}$$

be the characteristic polynomial corresponding to the recurrence $\{w_n\}$ with characteristic roots $r_1, r_2, \ldots, r_k$. By a classical result in the theory of finite differences,

$$w_n = \sum_{i=1}^{k} (c_i^{(0)} + c_i^{(1)}n + \cdots + c_i^{(m_i-1)}n^{(m_i-1)})r_i^n, \tag{2}$$

where the $c_i^{(j)}$ are complex constants. By (2),

$$s_n = w_{nt+d} = \sum_{i=1}^{k} (c_i^{(0)} + c_i^{(1)}n + \cdots + c_i^{(m_i-1)}n^{(m_i-1)})r_i^{nt+d}$$

$$= \sum_{i=1}^{k} (c_i^{(0)}r_i^d + c_i^{(1)}r_i^d n + \cdots + c_i^{(m_i-1)}r_i^d n^{(m_i-1)})(r_i^t)^n. \tag{3}$$

Since the roots r_i, $1 \leqslant i \leqslant k$, satisfy a monic polynomial over the integers, it follows that all the algebraic conjugates of a fixed characteristic root r_j appear among the r_i's. It then follows that all the algebraic conjugates of r_j^t appear among the t^{th} powers of the characteristic roots. Thus, $r_1^t, r_2^t, \ldots, r_k^t$ are the roots of a k^{th}-order integral monic polynomial

$$g(x) = x^k - a_1^{(t)}x^{k-1} - \cdots - a_{k-1}^{(t)}x - a_k^{(t)}. \tag{4}$$

It is evident that the root r_i^t of $g(x)$ appears with a multiplicity of at least m_i and that r_i^t satisfies the k^{th}-order linear integral recurrence

$$h_{n+k} = a_1^{(t)}h_{n+k-1} + a_2^{(t)}h_{n+k-2} + \cdots + a_k^{(t)}h_n \tag{5}$$

for $1 \leqslant i \leqslant k$. It is known and easily verified that if $P_i^{(m_i-1)}$ is a complex polynomial of degree at most $m_i - 1$, then the sequence $\{c_n\}$ defined by

$$c_n = \sum_{i=1}^{k} P_i^{(m_i-1)}(n)r_i^t$$

also satisfies the recursion relation given by (5). It thus follows from (3) that $\{s_n\} = \{w_{nt+d}\}$ also satisfies the same recursion relation.

It follows from (4) that, for $1 \leqslant j \leqslant k$,

$$-a_j^{(t)} = \sum (-1)^j r_{i_1}^t r_{i_2}^t \cdots r_{i_j}^t, \tag{6}$$

where one sums over all indices $i_1, i_2, \ldots, i_j$ such that

$$1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant k.$$

Thus, the coefficients $a_i^{(t)}$, $1 \leqslant i \leqslant k$, clearly depend on t but not on d.

Finally, it follows from (1) that

$$-a_k = (-1)^k r_1 r_2 \cdots r_k. \tag{7}$$

Thus, from (6) and (7), we see that

$$a_k^{(t)} = (-1)^{k+1}r_1^t r_2^t \cdots r_k^t = (-1)^{k+1}(r_1 r_2 \cdots r_k)^t$$

$$= (-1)^{k+1}[(-1)^{k+1}a_k]^t = (-1)^{(k+1)(t+1)}a_k^t.$$

We are now done.

Also solved by P. Bruckman, L. Dresel, and S. Papastavridis.

A Prime Result

H-378 *Proposed by M. Wachtel, Zurich, Switzerland*
(Vol. 22, no. 4, November 1984)

For every positive integer x and y, provided they are prime to each other, show that no integral divisor of $x^2 - 5y^2$ is congruent to 3 or 7, modulo 10.

Solution by J. M. Metzger, Grand Forks, ND

Let p be a prime divisor of $x^2 - 5y^2$. Now p is not a divisor of y for if so it divides x as well, contrary to the assumption that x and y are prime to each other. Since p does not divide y, y has a multiplicative inverse, say z, modulo p. So, from $x^2 - 5y^2 \equiv 0 \pmod{p}$, it follows that $(xz)^2 \equiv 5 \pmod{p}$. Thus, 5 is a quadratic residue modulo p. Hence $p = 2$, 5 or $p \equiv \pm 1 \pmod{10}$. Products of such primes can never be 3 or 7 modulo 10, and so $x^2 - 5y^2$ cannot have divisors congruent to 3 or 7 modulo 10.

Also solved by P. Bruckman, L. Dresel, L. Kuipers, L. Somer, T. White, and the proposer.

Sum Formula!

H-379 *Proposed by Andreas N. Philippou and Frosso S. Makri,*
University of Patras, Patras, Greece
(Vol. 22, no. 4, November 1984)

For each fixed integer $k \geqslant 2$, let $\{f_n^{(k)}\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k [1]. Show that

$$f_{n+2}^{(k)} = \sum_{i=0}^{\infty} \sum_{j=0}^{n} (-1)^i \binom{n - ik}{n - j} \binom{n - j + 1}{i}, \quad n \geqslant 0.$$

Reference

1. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order K." *The Fibonacci Quarterly* **20**, no. 1 (1982):28-32.

Solution by the proposers

The problem is trivially true for $n = 0$. It suffices therefore to show it for $n \geqslant 1$. Denote by S_n and L_n, respectively, the number of successes and the length of the longest success run in n ($\geqslant 1$) Bernoulli trials. It has been shown in [1] and [3] that

$$P[L_n \leqslant k - 1 | S_n = j] = \binom{n}{j}^{-1} \sum_{i=0}^{\infty} (-1)^i \binom{n - ik}{n - j} \binom{n - j + 1}{i} \tag{1}$$

and

$$P[L_n \leqslant k - 1 | S_n = j] = \binom{n}{j}^{-1} \sum_{i=0}^{k-1} \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}, \tag{2}$$

where the inner sum is taken over all nonnegative integers $n_1, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n - i$ and $n_1 + \cdots + n_k = n - j$. Relations (1) and (2) give

$$\sum_{j=0}^{n}\sum_{i=0}^{k-1}\sum_{\substack{n_1,\ldots,n_k\\ n_1+2n_2+\cdots+kn_k=n-i\\ n_1+\cdots+n_k=n-j}}\binom{n_1+\cdots+n_k}{n_1,\ldots,n_k}=\sum_{i=0}^{\infty}\sum_{j=0}^{n}(-1)^i\binom{n-ik}{n-j}\binom{n-j+1}{i}. \tag{3}$$

Now let $p = 1/2$. Then,

$$P[L_n \leqslant k-1] = \frac{1}{2^n}\sum_{i=0}^{k-1}\sum_{\substack{n_1,\ldots,n_k\\ n_1+2n_2+\cdots+kn_k=n-i}}\binom{n_1+\cdots+n_k}{n_1,\ldots,n_k}, \quad \text{by [2]},$$

$$= \frac{1}{2^n}\sum_{i=0}^{k-1} f_{n-i+1}^{(k)} = f_{n+2}^{(k)}/2^n, \quad \text{by [4]}. \tag{4}$$

Moreover,

$$P[L_n \leqslant k-1] = \sum_{j=0}^{n} P[L_n \leqslant k-1, S_n = j] = \sum_{j=0}^{n} P[L_n \leqslant k-1 | S_n = j]P[S_n = j]$$

$$= \frac{1}{2^n}\sum_{j=0}^{n}\sum_{i=0}^{k-1}\sum_{\substack{n_2,\ldots,n_k\\ n_1+2n_2+\cdots+kn_k=n-i\\ n_1+\cdots+n_k=n-j}}\binom{n_1+\cdots+n_k}{n_1,\ldots,n_k}, \quad \text{by (2)}. \tag{5}$$

The last three relations give

$$f_{n+2}^{(k)} = \sum_{i=0}^{\infty}\sum_{j=0}^{n}(-1)^i\binom{n-ik}{n-j}\binom{n-j+1}{i}, \quad n \geqslant 1,$$

which completes the proof of the problem.

References

1. E. J. Burr & G. Cane. "Longest Run of Consecutive Observations Having a Special Attribute." *Biometrika 48* (1961):461-465.
2. A. N. Philippou & F. S. Makri. "Longest Success Runs and Fibonacci-Type Polynomials." *The Fibonacci Quarterly 23*, no. 4 (1985):338-346.
3. A. M. Philippou & F. S. Makri. "Successes, Runs and Longest Runs." Submitted for publication.
4. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order K." *The Fibonacci Quarterly 20*, no. 1 (1982):28-32.

Also solved by P. Bruckman, C. Georghiou, and S. Papastavridis.

A Sparse Sequence

H-380 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC (Vol. 22, no. 4, November 1984)*

The sequence 1, 4, 5, 9, 13, 14, 16, 25, 29, 30, 36, 41, 49, 50, 54, 55, ... of squares and sums of consecutive squares appeared in Problem B-495. Show that this sequence has Schnirelmann density zero.

Solution by Paul S. Bruckman, Fair Oaks, CA

Let S denote the given sequence. We may characterize S as the sequence of sums of the form

$$q(i, j) = \sum_{k=i}^{i+j-1} k^2, \quad i, j \geqslant 1. \tag{1}$$

Given n, let $P(n)$ denote the number of pairs (i, j) such that $q(i, j) \leqslant n$. Then $\lim_{n\to\infty} P(n)/n$ is the Schnirelmann density of S, which we seek to prove is zero.

Now $q(i, j) = q(1, i + j - 1) - q(1, i - 1)$; after some simplification, we find

$$q(i, j) = ji(i + j - 1) + q(1, j - 1). \tag{2}$$

Assuming j fixed for the time being, we see from (2) that $q(i, j) \leqslant n$ implies $ji^2 \leqslant n$, or $i \leqslant (n/j)^{1/2}$; also,

$$j^3/3 < \frac{1}{6} j(j + 1)(2j + 1) = q(1, j) \leqslant q(i, j) \leqslant n,$$

so $j^3/3 < n$, or $j < (3n)^{1/3}$. Therefore,

$$P(n) \leqslant \sum_{j=1}^{m} (n/j)^{1/2}, \quad \text{where } m = [(3n)^{1/3}]. \tag{3}$$

Now consider the sum

$$Z(N) = \sum_{k=1}^{N} k^{-1/2}, \quad \text{where } N \text{ is large.} \tag{4}$$

We see that

$$Z(N) = N^{1/2} \sum_{k=1}^{N} (k/N)^{-1/2} N^{-1} \sim N^{1/2} \int_0^1 x^{-1/2}\, dx \text{ as } N \to \infty.$$

Since

$$\int_0^1 x^{-1/2}\, dx = 2x^{1/2} \Big]_0^1 = 2,$$

thus

$$Z(N) = 0(N^{1/2}) \text{ as } N \to \infty. \tag{5}$$

Returning to (3), we see that $P(n) \leqslant Q(n)$, where $Q(n) = 0(n^{1/2} \cdot m^{1/2}) = 0(n^{2/3})$ as $n \to \infty$; hence

$$P(n) = 0(n^{2/3}) \text{ as } n \to \infty.$$

Thus, $P(n)/n = 0(n^{-1/3}) = 0(n)$ as $n \to \infty$. Q.E.D.

Also solved by C. Georghiou and the proposer.

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