A RELATION FOR THE PRIME DISTRIBUTION FUNCTION

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In this paper, we obtain an interesting duality relationship between the prime distribution function (π-function) and another, less well-known, number theoretic function. The domain of definition throughout is the set of natural numbers.

We recall the definition of the π-function:

$$\pi(n) = \sum_{p \leqslant n} 1, \text{ which counts the number of primes} \leqslant n. \tag{1}$$

Also, we recall the Möbius function, defined as follows:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n \text{ is divisible by a square (or higher power)} \\ & \text{of a prime;} \\ (-1)^k, & \text{if } n \text{ is the product of } k \text{ distinct primes.} \end{cases} \tag{2}$$

We also indicate, without proof, a well-known relationship satisfied by the Möbius function:

$$\sum_{d|n} \mu(d) = \delta_{1n} = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1, \end{cases} \tag{3}$$

where the sum is taken over all divisors d of n.

We now introduce another function $\lambda(n)$ which seeks to enumerate all *powers* of primes (including first powers) so that such powers are $\leqslant n$. We may count $\lambda(n)$ by letting k vary from $1, 2, 3, \ldots$ and counting the acceptable k^{th} powers of primes. For a given prime p, the inequality $p^k \leqslant n$ is equivalent to

$$k \leqslant \frac{\log n}{\log p},$$

and is satisfied by

$$k = 1, 2, 3, \ldots, \left[\frac{\log n}{\log p}\right], \text{ i.e., for } \left[\frac{\log n}{\log p}\right] \text{ values.}$$

Summing over all p, we thus obtain:

$$\lambda(n) = \sum_{p \leqslant n} \left[\frac{\log n}{\log p}\right]. \tag{4}$$

An alternative expression for $\lambda(n)$ can be obtained by noting that $p^k \leqslant n$ is equivalent to $p \leqslant [n^{1/k}]$. The component of $\lambda(n)$ that counts all k^{th} powers of primes thus counts all primes $p \leqslant [n^{1/k}]$, and must therefore equal $\pi([n^{1/k}])$. Summing over all possible k, we therefore obtain the relationship:

$$\lambda(n) = \sum_{k=1}^{\infty} \pi([n^{1/k}]). \tag{5}$$

Note that the "infinite" series in (5) actually terminates since, for sufficiently large k, $[n^{1/k}] = 1$ for all n, and $\pi(1) = 0$.

The relationship in (5) may be inverted to yield an expression for $\pi(n)$ in terms of λ, valued at varying arguments. This expression is as follows:

$$\pi(n) = \sum_{k=1}^{\infty} \mu(k)\lambda([n^{1/k}]). \tag{6}$$

A comment similar to that following (5) applies here, too, since $\lambda(1) = 0$.

To prove (6), we resort to a pair of seemingly unrelated lemmas.

Lemma 1: Given positive integers m, n, and r, let

$$\chi(m|n) = \begin{cases} 1, & \text{if } m|n; \\ 0, & \text{if } m \nmid n. \end{cases}$$

Define $r \times r$ matrices $A_r = (a_{ij}^{(r)})$ and $B_r = (b_{ij}^{(r)})$ as follows:

$$a_{ij}^{(r)} = \chi(i|j); \tag{8}$$

$$b_{ij}^{(r)} = \chi(i|j)\mu(j/i), \quad (i, j = 1, 2, 3, \ldots, r). \tag{9}$$

Then

$$A_rB_r = I_r, \text{ i.e., } B_r = A_r^{-1}. \tag{10}$$

Proof of Lemma 1: Let $A_rB_r = C_r = (c_{ij}^{(r)})$. Then

$$c_{ij}^{(r)} = \sum_{k=1}^{r} a_{ik}^{(r)}b_{kj}^{(r)} = \sum_{k=1}^{r} \chi(i|k)\chi(k|j)\mu(j/k).$$

Note that each term of this sum vanishes *unless* $i|k|j$, i.e., unless $i|j$. Thus, $c_{ij}^{(r)} = 0$ if $i \nmid j$. Suppose now that $i|j$, and let $j = id$. Then

$$c_{ij}^{(r)} = \sum_{u=1}^{j/i} \chi(ui|j)\mu(j/ui) = \sum_{u=1}^{d} \chi(u|d)\mu(d/u) = \sum_{d_1|d} \mu(d_1) = \delta_{1d}$$

[using (3)]. Hence,

$$c_{ij}^{(r)} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

This is equivalent to $C_r = I_r$. Q.E.D.

Lemma 2: Suppose n, a, and b are positive integers. Then

$$[[n^{1/a}]^{1/b}] = [n^{1/ab}]. \tag{11}$$

Proof of Lemma 2: Let $u = [n^{1/a}]$. Since $n^{1/a} \geqslant 1$, thus $u \geqslant 1$. Define the integer $v \geqslant 2$ by:

$$1 \leqslant (v - 1)^b \leqslant u < v^b - 1.$$

Since $n^{1/a} < u + 1 < v^b$, thus $n^{1/ab} < v$, which implies $[n^{1/ab}] \leqslant v - 1$. However, $v - 1 \leqslant u^{1/b} < (v^b - 1)^{1/b} < v$, which implies $[u^{1/b}] = v - 1$; therefore,

$$[n^{1/ab}] \leqslant [u^{1/b}]. \tag{12}$$

On the other hand, $n^{1/a} \geqslant u \Rightarrow n^{1/ab} \geqslant u^{1/b}$, which implies

$$[n^{1/ab}] \geqslant [u^{1/b}]. \tag{13}$$

It follows from (12) and (13) that

$$[n^{1/ab}] = [u^{1/b}], \tag{14}$$

which is equivalent to (11). Q.E.D.

The proof of (6) follows. Let $f(k) = \pi([n^{1/k}])$, $g(k) = \lambda([n^{1/k}])$, assuming n is given. Applying Lemma 2 and (5) indefinitely, with n replaced successively by $[n^{1/r}]$, $r = 1, 2, 3, \ldots$, the following relationships are evident:

$$g(r) = \sum_{k=1}^{\infty} f(rk), \quad r = 1, 2, 3, \ldots\ . \tag{15}$$

Let us define the following vectors:

$$\mathbf{f}_r' = (f(1), f(2), \ldots, f(r)), \quad \mathbf{g}_r' = (g(1), g(2), \ldots, g(r)). \tag{16}$$

We may then transform (15) into matrix notation as follows:

$$\mathbf{g}_r = A_r \mathbf{f}_r. \tag{17}$$

Multiplying both sides of (17) by B_r, as given in Lemma 1, yields the desired inversion formula:

$$\mathbf{f}_r = B_r \mathbf{g}_r. \tag{18}$$

Converting (18) back to scalar notation, we obtain:

$$f(r) = \sum_{k=1}^{\infty} \mu(k) g(rk). \tag{19}$$

Now, setting $r = 1$ in (19) yields the desired result in (6). Q.E.D.

Lemma 1 is a very interesting result in its own right, and it provides the basis for the well-known technique of Möbius inversion, of which the dual relationships given in (5) and (6) are special cases.

Note that (6) provides an explicit expression for the prime distribution function, which is an important step in one of the most celebrated of unsolved problems in number theory, namely the discovery of an explicit formula for the n^{th} prime. Before giving vent to undue jubilation, however, it must be noted that the "explicit" expression given by (6) is in terms of another auxiliary number theoretic function, which is itself not readily found in terms of n. Therefore, the pair of relationships in (5) and (6) is apparently only of academic interest insofar as the great unsolved problem is concerned. It may come to pass, nevertheless, that some reader of this paper will find some use for these relationships toward the solution of this or other problem.

We conclude this paper with a brief table of the first few values of the two functions studied herein.

n	$\pi(n)$	$\lambda(n)$
1	0	0
2	1	1
3	2	2
4	2	3
5	3	4
6	3	4
7	4	5
8	4	6
9	4	7
10	4	7
11	5	8
12	5	8
13	6	9
14	6	9
15	6	9
16	6	10
17	7	11
18	7	11
19	8	12
20	8	12
21	8	12
22	8	12
23	9	13
24	9	13

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