SOME COMBINATORIAL SEQUENCES

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1. INTRODUCTION

We will enumerate the different $m \times m$ matrices $B_r(n)$, $n = 1, 2, 3, \ldots, r = 1, 2, 3, \ldots, x_n$, having elements from the set [0, 1], where the allowed column vectors B_j and some conditions between elements b_{ij} are specified. That is,

$$\text{C1:} \quad b_{ij} = 1 \Rightarrow b_{i,j-1} = 0,$$

$$\text{C2:} \quad b_{ij} = 1 \Rightarrow \begin{cases} b_{i-1,j} = 0 \\ b_{i+1,j} = 0, \; m > i > 1, \end{cases}$$

and

$$b_{1j} = 1 \Rightarrow b_{2j} = 0,$$

$$b_{mj} = 1 \Rightarrow b_{m-1,j} = 0.$$

The number of different matrices $B_r(n)$ is called x_n and is the general term of a combinatorial sequence $\{x_n : n = 1, 2, 3, \ldots\}$. The vector $B_j = P_j$ is one of the p distinct column vectors in an $m \times p$ matrix P called the primitive matrix. The vector P_j is named in accordance with the following rules:

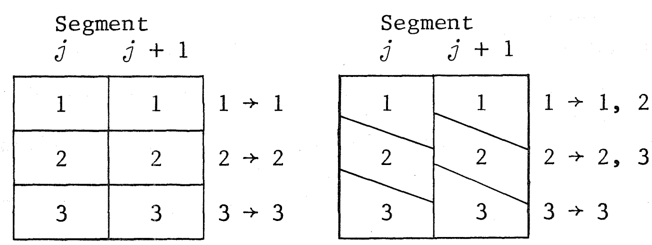
1. The name of the zero vector is 0; the remaining vectors may be identified by the positions of 1's in them.
2. The numbers in these names, if more than one, are conveniently given in increasing order with a bar placed over them.
3. The dimension m of B_j is greater than or equal to the largest number in its name.

EXAMPLES

Name of P_j	P_j
0	0 0 0 ... 0
1	1 0 0 ... 0
2	0 1 0 ... 0
$\overline{12}$	1 1 0 ... 0
$\overline{13}$	1 0 1 0 ... 0
$\overline{123}$	1 1 1 0 ... 0

m	Some Primitive Matrices P	
	Under C2	Unrestricted
1	(0 1)	(0 1)
2	(0 1 2)	(0 1 $\overline{12}$ 2)
3	(0 1 $\overline{13}$ 2 3)	(0 1 $\overline{12}$ $\overline{123}$ $\overline{13}$ 2 $\overline{23}$ 3)
4	(0 1 $\overline{13}$ $\overline{14}$ 2 $\overline{24}$ 3 4)	(0 1 $\overline{12}$ $\overline{123}$ $\overline{1234}$ $\overline{124}$ $\overline{13}$ $\overline{134}$ $\overline{14}$ 2 $\overline{23}$ $\overline{234}$ $\overline{24}$ 3 $\overline{34}$ 4)
Size	$m \times F_{m+2}$	$m \times 2^m$

Any figure consisting of a succession of like segments each of which is divided into m cells which can be occupied by either a 1 or a 0 under given conditions may be represented by a matrix $B_r(n)$ in which n is the number of segments in the figure. The cells in any segment must be numbered in a given way (1, 2, 3, ..., m) and correspond to the row numbers in $B_r(n)$. Figures in which only cells of like number in adjacent segments are adjacent are said to be *regular*. This adjacency condition (AC) is symbolized by $b_i \rightarrow b_i$. Figures in which at least one cell b_{ij} in the j^{th} segment is adjacent to more than one cell in the $(j+1)^{\text{st}}$ segment $(b_{s,j+1}, b_{t,j+1}, \ldots)$ are said to be *irregular*. This AC is symbolized by $b_i \rightarrow b_s, b_t, \ldots$ (see Fig. 1).



(a) Regular Figure

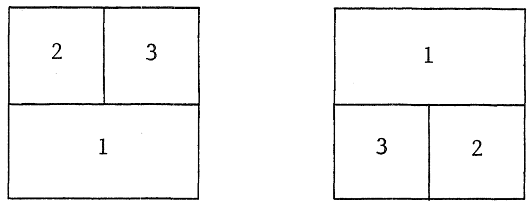
(b) Irregular Figure

Figure 1

Consider a prism of n segments formed of segments of unit height on bases A or B (Figure 2). If the segments have equal bases A or B, P = (0 1 2 3) is a possible primitive matrix and $b_i \rightarrow b_i$. If the successive segments have bases that alternate between A and B, P may be unchanged but $1 \rightarrow 2, 3$; $2 \rightarrow 1$; $3 \rightarrow 1$.

Condition 1 may be replaced by the more general condition C3: any two adjacent cells, each from a different segment cannot both contain the number 1.

The matrix P has a companion matrix $\overline{P}$ in which the column P_j has a counterpart $\overline{P}_j$ in $\overline{P}$ obtained by applying the given AC, $b_i \rightarrow b_s, b_t, \ldots$, to each number i in the name of P_j and ordering the resulting numbers without repetition. A bar is placed over these numbers to distinguish the columns of $\overline{P}$. That is, if P = (1 $\overline{12}$ $\overline{13}$ 2 3) in Figure 1(b), then $\overline{P}$ = ($\overline{12}$ $\overline{123}$ $\overline{123}$ $\overline{23}$ 3).



Base A Base B

Figure 2

Define the $(p+1) \times 1$ set matrix $M(1)$ with elements consisting of sets of matrices such that $m_1(1) = \emptyset$, the empty set, and $m_i(1) = [P_{i-1}]$, where $p + 1 \geqslant i > 1$ and L is the $(p+1) \times (p+1)$ partitioned matrix

$$L = \left(\begin{array}{c|c} \multicolumn{2}{c}{U^T} \\ \hline 0 & K \end{array}\right),$$

where 0 is the $p \times 1$ zero vector, U is $(p+1) \times 1$ with $u_1 = 0$, and $u_i = 1$ if $p + 1 \geqslant i > 1$. A matrix K, called the kernel, is $p \times p$ with $K_{ij} \in [0, 1]$ and is a function of P and the given AC as described later.

A special product is defined for L and a conforming set matrix generating another set matrix as a product.

$$L \cdot M(n-1) = M(n),\ n > 1, \tag{1.1}$$

hence

$$(L.)^{n-1}M(1) = M(n). \tag{1.2}$$

The expression $\ell_{ji}m_i(n-1)(P_{j-1})$ represents the result of augmenting each member of the set $m_i(n-1)$ by appending the vector P_{j-1} on the right if $\ell_{ji} = 1$. If $\ell_{ji} = 0$, this expression represents $\emptyset$.

$$m_1(n) = \bigcup_2^{p+1} \ell_{1i}m_i(n-1)$$

$$m_j(n) = \bigcup_2^{p+1} \ell_{ji}m_i(n-1)(P_{j-1}),\ j > 1.$$

Define $N(1)$ as the vector with $n_1(1) = 0$ and $n_j(1) = 1$ if $p + 1 \geqslant j > 1$. Let

$$LN(n-1) = N(n),\ n > 1. \tag{1.3}$$

The sets $m_j(n)$, $p + 1 \geqslant j > 1$ are disjoint, and their cardinality is unchanged by appending columns to their matrix elements. It can be shown by mathematical induction that $N(n)$ is a vector with $n_1(n) = x_{n-1}$ and that $n_j(n)$ is the number of matrices $B_r(n)$ having P_{j-1} for the n^{th} column.

Let $\mathbf{N}_n$ be the $p \times 1$ matrix with $\mathbf{n}_i(n) = n_{i+1}(n)$, $p \geqslant i > 1$, then

$$x_n = n_1(n+1) = \sum_2^{p+1} n_i(n) = \sum_1^{p} \mathbf{n}_i(n). \tag{1.4}$$

Example: Let $B_r(n)$ represent a $2 \times n$ matrix with $P = [0 \quad 1]$. If C1 holds,

$$k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad M(1) = \begin{bmatrix} \emptyset \\ [0] \\ [1] \end{bmatrix}, \quad N(1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{ and } L = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$LM(n-1) = M(n) \qquad \text{and} \qquad LN(n-1) = N(n),$$

so

$$M(1) = \begin{bmatrix} \emptyset \\ [0] \\ [1] \end{bmatrix}, \qquad N(1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

$$M(2) = \begin{bmatrix} [0,\ 1] \\ [00,\ 10] \\ [01] \end{bmatrix}, \qquad N(2) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad x_1 = 2,$$

$$M(3) = \begin{bmatrix} [00,\ 10,\ 01] \\ [000,\ 100,\ 010] \\ [001,\ 101] \end{bmatrix}, \qquad N(3) = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}, \quad x_2 = 3,$$

$$x_n = F_{n+2}.$$

Equation (1.3) implies

$$K\mathbf{N}(n) = \mathbf{N}(n+1) \tag{1.5}$$

so

$$K^n\mathbf{N}(1) = \mathbf{N}(n+1). \tag{1.6}$$

Let kernel K_r yield a value $n_1(n+1) = x_{rn}$, then if K_1 and K_2 yield $x_{1n} = x_{2n}$ they are said to be *virtually equivalent* and $K_1 \approx K_2$. Virtual equivalence is an equivalence relation.

Let Q_r represent a $p \times p$ permutation matrix, i.e., a square matrix whose elements in any row or column are all zero except for one element which is one. There are $p!$ such matrices and since $Q_rQ_r^T = I$, $Q_r^T = Q_r^{-1}$. From Equation (1.6), $K^{n-1}\mathbf{N}(1) = \mathbf{N}(n)$ and, if K is replaced by $Q_rKQ_r^{-1}$,

$$(Q_rKQ_r^{-1})^{n-1}\mathbf{N}(1) = Q_rK^{n-1}Q_r^T\mathbf{N}(1) = Q_rK^{n-1}\mathbf{N}(1) = Q_r\mathbf{N}(n).$$

From Equation (1.4), $x_n = \sum_1^p \mathbf{n}_i(n)$ for K and for $Q_rKQ_r^T$; the $\mathbf{n}_i(n)$ are summed in possibly a different order. The result is the same, so

$$Q_rKQ_r^T \approx K. \tag{1.7}$$

Let K_r be a $p_r \times p_r$ kernel, $r = 1, 2, 3$, and define the direct sum

$$K_1 \oplus K_2 = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

Permutation matrices Q_s and Q_s^T can be constructed so that

$$Q_s(K_1 \oplus K_2)Q_s^T = K_2 \oplus K_1.$$

If $q_{ij} \in Q_s$, then $q_{ij} = 1$ if

i	1	2	$\cdots$	p_2	$p_2 + 1$	$p_2 + 2$	$\cdots$	$p_2 + p_1$
j	$p_1 + 1$	$p_1 + 2$	$\cdots$	$p_1 + p_2$	1	2	$\cdots$	p_1

and $q_{ij} = 0$ otherwise. Let $p_1 = 2$ and $p_2 = 3$, then

$$Q_s = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

From Equation (1.7),

$$K_1 \oplus K_2 \approx K_2 \oplus K_1. \tag{1.8}$$

Define the direct product $K_1 \times K_2$ as the partitioned matrix

$$K_1 \times K_2 = \begin{bmatrix} k_{111}K_2 & k_{112}K_2 & \cdots & k_{11p_1}K_2 \\ k_{121}K_2 & k_{122}K_2 & \cdots & k_{12p_1}K_2 \\ \cdots & \cdots & \cdots & \cdots \\ k_{1p_1 1}K_2 & k_{1p_1 2}K_2 & \cdots & k_{1p_1p_1}K_2 \end{bmatrix}$$

in which $k_{1rs} \in K_1$ and $k_{2tu} \in K_2$.

Let

$$k_{1rs}k_{2tu} = \begin{cases} k'_{iv} \in K_1 \times K_2 \\ k''_{jw} \in K_2 \times K_1 \end{cases},$$

then

$$i = (r - 1)p_2 + t \tag{a}$$

and

$$j = (t - 1)p_1 + r. \tag{b}$$

From Equation (a),

$$t - 1 = (i - 1) \bmod p_2 \tag{c}$$

and

$$r - 1 = \left[\frac{i - 1}{p_2}\right], \tag{d}$$

in which $[x]$ represents the greatest integer in the number x. Substituting Equations (c) and (d) in (b),

$$j = p_1((i - 1) \bmod p_2) + \left[\frac{i - 1}{p_2}\right] + 1. \tag{e}$$

If i, j, r, and t are replaced by v, w, s, and u, respectively, Equations (a)-(e) still hold and Equation (e) becomes

$$w = p_1((v - 1) \bmod p_2) + \left[\frac{v - 1}{p_2}\right] + 1. \tag{f}$$

Consider a matrix Q where $q_{ij} = 1$ if Equation (e) is satisfied and $q_{ij} = 0$ otherwise. From Equation (a) if i is given, r and t are uniquely defined, and from Equation (b) j is uniquely defined. Conversely, if j is given, then i is uniquely defined. This implies that every row and column of Q has just one element 1 and all other elements are zero. Q is then a permutation matrix.

Consider the matrix Q' where $q'_{vw} = 1$ if Equation (f) is satisfied and $q'_{vw} = 0$ otherwise. By a similar argument, Q' is also a permutation matrix and since j and i may replace w and v, respectively, in Equation (f) to produce Equation (e), then we let $Q_p = Q' = Q$ so that

$$Q_p(K_1 \times K_2)Q_p^T = K_2 \times K_1.$$

From Equation (1.7),

$$K_1 \times K_2 \approx K_2 \times K_1. \tag{1.9}$$

For example, if $p_1 = 2$ and $p_2 = 3$,

$$Q_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let

$$K_3 = K_1 \oplus K_2 = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \quad \text{and} \quad \mathbf{N}_3(1) = \begin{bmatrix} \mathbf{N}_1(1) \\ \mathbf{N}_2(1) \end{bmatrix}$$

then

$$K_3^{n-1} = \begin{bmatrix} K_1^{n-1} & 0 \\ 0 & K_2^{n-1} \end{bmatrix}$$

and, by Equation (1.6)

$$\mathbf{N}_3(n) = \begin{bmatrix} \mathbf{N}_1(n) \\ \mathbf{N}_2(n) \end{bmatrix}$$

Applying Equation (1.4),

$$x_{3n} = x_{1n} + x_{2n} \quad \text{if } K_3 = K_1 \oplus K_2. \tag{1.10}$$

Suppose $K_3 = K_1 \times K_2$ with $\mathbf{N}_3(1) = \mathbf{N}_1(1) \times \mathbf{N}_2(1)$, a $p_1p_2 \times 1$ matrix of 1's. Then, by Equation (1.4), $x_{31} = x_{11}x_{21}$. Assume that $\mathbf{N}_3(r) = \mathbf{N}_1(r) \times \mathbf{N}_2(r)$ for any $r > 0$, then

$$K_3\mathbf{N}_3(r) = (K_1 \times K_2)(\mathbf{N}_2(r) \times \mathbf{N}_1(r)) = \sum_{j=1}^{p_1} k_{1ij}n_{1ji}(r)K_2\mathbf{N}_2(r), \; i = 1, 2, \ldots, p_1,$$

or $K_3N_3(r) = K_1N_1(r) \times K_2N_2(r)$, and by Equation (1.5),

$$N_3(r + 1) = N_1(r + 1) \times N_2(r + 1).$$

It follows by mathematical induction that $N_3(n) = N_1(n) \times N_2(n)$ for all n and, from Equation (1.4),

$$x_{3n} = x_{1n}x_{2n} \quad \text{if } K_3 = K_1 \times K_2. \tag{1.11}$$

From definitions

$$(K_1 \oplus K_2) \times K_3 = (K_1 \times K_3) \oplus (K_2 \times K_3), \tag{1.12}$$

32 virtual equivalences may be deduced using the commutative laws for $\oplus$ and $\times$.

2. EVALUATION OF K

Theorem 2.1: If C3 holds, and if $\overline{P}_i$ and P_j have one or more numbers common in their names, then $k_{ij} = 0$; if $\overline{P}_i$ and P_j have no numbers common in their names, then $k_{ij} = 1$.

Proof: From Equation (1.1), $L \cdot M(n - 1) = M(n)$, and by renumbering elements,

$$\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & k_{11} & k_{12} & \cdots & k_{1p} \\ 0 & k_{21} & k_{22} & \cdots & k_{2p} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & k_{p1} & k_{p2} & \cdots & k_{pp} \end{bmatrix} \begin{bmatrix} m_0(n-1) \\ m_1(n-1) \\ m_2(n-1) \\ \vdots \\ m_p(n-1) \end{bmatrix} = \begin{bmatrix} m_0(n) \\ m_1(n) \\ m_2(n) \\ \vdots \\ m_p(n) \end{bmatrix}.$$

Through multiplication,

$$m_0(n) = \bigcup_1^p m_i(n-1)(\emptyset) = \bigcup_1^p m_i(n-1),$$
$$m_j(n) = \bigcup_1^p k_{ji}m_i(n-1)(P_j), \quad j = 1, 2, \ldots, p,$$

where $m_i(n - 1)(P_j)$ represents the set $m_i(n - 1)$ in which each element $B_r(n - 1)$ has P_i as the terminal column and is augmented by the vector P_j to form a matrix $B_r'(n)$. The last two columns of $B_r'(n)$ are P_i and P_j. If P_i has one or more elements of value one adjacent to a like element in P_j, the name of $\overline{P}_i$ must have one or more numbers in common with the name of P_j, and C3 implies $B_r'(n) \notin m_j(n)$, hence $K_{ij} = 0$. If P_i has no elements of value one adjacent to a like element in P_j, the name of $\overline{P}_i$ and the name of P_j must have no numbers in common and C3 implies $B_r'(n) \in m_j(n)$, so $K_{ij} = 1$. ■

Let $R = \overline{P}_i^T P_j = (r_{11})$, a 1×1 matrix. Then

Corollary 2.1: If C3 holds and $r_{11} = 0$, $k_{ij} = 1$; if $r_{11} > 0$, $k_{ij} = 0$.

Corollary 2.2: If C1 holds, K is symmetric.

Proof: If C1 holds, $P_i = \overline{P}_i$, so $R = (r_{11}) = R^T$ and $\overline{P}_i^T P_j = P_i^T P_j = P_j^T P_i = \overline{P}_j^T P_i$. By Corollary 2.1, if $r_{11} = 0$, $k_{ij} = k_{ji} = 1$; if $r_{11} > 0$, $k_{ij} = k_{ji} = 0$. Since

$r_{11} \geqslant 0$, $k_{ij} = k_{ji}$ and K is symmetric. ∎

Corollary 2.3: If C3 holds and $P_i = 0$, then $k_{ij} = 1$ for all j; $P_i \neq 0$ implies $k_{ii} = 0$.

Corollary 2.4: If C3 holds, then K can have at most one row of 1's.

Let $X = [x_i : i = 1, 2, 3, \ldots, r]$, $m \geqslant r > 0$, be the set of all the different numbers appearing in the names of the columns of P and in the AC, and let $Y = [y_i =: i = 1, 2, 3, \ldots, r]$ be any other set of r distinct numbers, then

Corollary 2.5: K is unchanged by replacing x_i by y_i, $i = 1, 2, 3, \ldots, r$, in P and in the AC under C3.

Definition: A proper K is a K in which there is at most one row of 1's.

Theorem 2.2: Every proper K may be derived from some P under $C3$ and AC.

Proof: Given $k_{ij} \in [0, 1]$. If a row K_i consists only of 1's, it is named 0 and the remaining rows are named 1, 2, 3, ..., $p - 1$. If no such row exists, name the rows 1, 2, 3, ..., p. Then P consists of columns P_j which are in the same sequence as the named rows of K and have the same names. Suppose K_i has an element $k_{ij} = 0$, then the AC must include $i \to j$; if $k_{ij} = 1$, then $i \not\to j$. Since K is proper, there is at most one row of 1's which is named 0. All columns of P have names which are unique. ∎

The AC under C3 may sometimes be simplified by changing the columns of P without altering K. Let d, e, and f represent three distinct cells in a segment B_j of $B_r(n)$ and let r, s, and $r \cup s$ represent sets of cells in B_{j+1} adjacent to d, e, and f, respectively. The adjacency conditions are represented by the set $[d \to r,\ e \to s,\ f \to r \cup s]$, and f may be replaced by $\overline{de}$ in the names of P_j and in AC forming P' and the set $\overline{AC} = [d \to r,\ c \to s]$ which, by Theorem 2.1, yields the same K.

Example: Let

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

By Theorem 2.2, K may be derived from $P = (1\ 2\ 3\ 4\ 5\ 6\ 7)$ under C3 with the AC

$1 \to 1, 2, 3, 4$
$2 \to 1, 2, 3, 4, 5, 6$
$3 \to 1, 2, 3, 4, 5, 6, 7$
$4 \to 1, 2, 3, 4, 6, 7$
$5 \to 2, 3, 5, 6$
$6 \to 2, 3, 4, 5, 6, 7$
$7 \to 3, 4, 6, 7.$

The AC may be simplified as follows:

Consider:

$$1 \to 2,\ 3,\ 1,\ 4$$
$$2 \to 2,\ 3,\ 1,\ 4,\ 5,\ 6$$
$$5 \to 2,\ 3,\ 5,\ 6$$

We can then replace 2 by $\overline{15}$. Similarly we can replace 3 by $\overline{24}$, 4 by $\overline{17}$, and 6 by $\overline{57}$, so P becomes $P' = (1\ \ \overline{15}\ \ \overline{157}\ \ \overline{17}\ \ 5\ \ \overline{57}\ \ 7)$. By renumbering in accordance with Corollary 2.5, $P' = (1\ \ \overline{12}\ \ \overline{123}\ \ \overline{13}\ \ 2\ \ \overline{23}\ \ 3)$ with

$$AC = [1 \to 1;\ 2 \to 2;\ 3 \to 3].$$

Further examples giving P, AC, K, x_n, and recurrence relations are:

#1 $P = (0\ \ 1)$
$AC = [1 \to 1]$
$x_n = \{2, 3, 5, 8, 13, \ldots\}$
$x_{n+2} - x_{n+1} - x_n = 0$

$$K = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

#2 $P = (0\ \ 1\ \ 2)$
$AC = [1 \to 1;\ 2 \to 2]$
$x_n = \{3, 7, 17, 41, 99, 239, \ldots\}$
$x_{n+2} - 2x_{n+1} - x_n = 0$

$$K = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

#3 $P = (1\ \ 2)$
$AC = [1 \to 1, 2;\ 2 \to 1, 2]$
$x_n = \{2, 0, 0, \ldots\}$
$x_{n+1} = 0$

$$K = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

#4 $P = (0\ \ 1\ \ 2)$
$AC = [1 \to 1, 2;\ 2 \to 1, 2]$
$x_n = \{3, 5, 11, 21, 43, 85, \ldots\}$
$x_{n+2} - x_{n+1} - 2x_n = 0$

$$K = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

#5 $P = (1\ \ 2\ \ 3\ \ 4\ \ 5)$
$AC = [1 \to 3, 4, 5;\ 2 \to 2, 3, 4, 5;$
$3 \to 1, 2;\ 4 \to 1, 2, 4;$
$5 \to 1, 2, 5]$
$x_n = \{5, 10, 22, 49, 112, 260, \ldots\}$
$x_{n+4} - 3x_{n+3} + 3x_{n+1} + x_n = 0$

$$K = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

#6 $P = (0\ \ 1\ \ 2\ \ 3)$
$AC = [1 \to 1, 3;\ 2 \to 2, 3;\ 3 \to 1, 2, 3$
also $P = (0\ \ 1\ \ 2\ \ \overline{12})$
$AC = [1 \to 1;\ 2 \to 2]$
$x_n = \{4, 9, 25, 64, 169, 441, \ldots\}$
$x_{n+4} - x_{n+3} - 4x_{n+2} - x_{n+1} + x_n = 0$

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Example #1 represents the sequence $x_n = F_{n+2}$. Examples #2 and #4 represent sequences of Winthrop and Horadam [2], $x_n = w_n(1, 3; 2, -1)$ and $x_n = w_n(1, 3; 1, -2)$, respectively, where $w\ (a, b; p, q)$ has $w_0 = a$, $w_1 = b$, and $w_n = pw_{n-1} - qw_{n-1}$, $n \geqslant 2$. Example #5 illustrates $K_3 = K_1 \oplus K_2$ with $x_n = F_{n+2} + w_n(1, 3; 2, -1)$, and Example #6 illustrates $K_3 = K_1 \times K_2$ with $x_n = (F_{n+2})^2$ in which two values for P and the corresponding AC are given.

3. RECURRENCE RELATIONS

The characteristic function of K is $f(y) = |yI - K|$ and its characteristic equation is

$$f(y) = \sum_0^p c_i y^i = 0. \tag{3.1}$$

Theorem 3.1:

$$\sum_0^p c_i x_{n+i} = 0$$

is a recurrence relation for the sequence $\{x_n : n = 1, 2, 3, \ldots\}$.

Proof: Apply the Cayley-Hamilton theorem to Equation (3.1), giving

$$\sum_0^p c_i K^i = 0.$$

Multiply each side of this on the right by $K^{n-1}\mathbf{N}(1)$, giving

$$\sum_0^p c_i K^{n-1+i}\mathbf{N}(1).$$

Then, by Equation (1.6),

$$\sum_0^p c_i \mathbf{N}(n + i) = 0.$$

Multiply on the left by $\mathbf{U}^T$, a $1 \times p$ matrix with $\mathbf{u}_{1i} = 1$, giving

$$\sum_0^p c_i \sum_0^p \mathbf{n}_j(n + i) = 0,$$

and by Equation (1.4),

$$\sum_0^p c_i x_{n+i} = 0.$$

This is a recurrence relation for the sequence $\{x_n : n = 1, 2, 3, \ldots\}$. ∎

Corollary 3.1: If the characteristic equation of K is

$$(y - d)\sum_0^{p-1} c_i y^i = 0$$

and if $K - dI$ is nonsingular, then

$$\sum_0^{p-1} c_i x_{n+i} = 0$$

is a recurrence relation for $\{x_n : n = 1, 2, 3, \ldots\}$.

Proof: By the Cayley-Hamilton theorem,

$$(K - dI)\sum_0^{p-1} c_i K^i = 0.$$

If $K - dI$ is nonsingular, apply its inverse to both sides of the equation, so

$$\sum_0^{p-1} c_i x^i = 0.$$

Proceed as in Theorem 3.1 to show that $\sum_0^{p-1} c_i x_{n+i} = 0$ is the desired recurrence relation. ■

Note that if $\mathbf{N}(1)$, in which $\mathbf{n}_{i1} = 1$, were defined as some other vector of size $p \times 1$, the new sequence $\{x_n\}$ would still possess the same recurrence relation.

Let

$$f_j(y) = \sum_0^{p_j} c_{jq} y^q = 0$$

represent the characteristic equation for $K_j : j = 1, 2, 3$.

Theorem 3.2: If $K_3 = K_1 \oplus K_2$, a recurrence relation for the sequence $\{x_{3n} : n = 1, 2, 3, \ldots\}$ is

$$\sum_0^{p_1+p_2} \sum_{q+r=i} c_{1q} c_{2r} x_{3(n+1)} = 0.$$

Proof: $$\sum_0^{p_3} c_{3i} y^i = \begin{vmatrix} yI - K_1 & 0 \\ 0 & yI - K_2 \end{vmatrix} = |yI - K_1||yI - K_2|$$

$$= \sum_0^{p_1} c_{1q} y^q \sum_0^{p_2} c_{2r} y^r,$$

then

$$c_{3i} = \sum_{q+r=i} c_{1q} c_{2r}$$

and, from Theorem 3.1, the recurrence relation for the sequence $\{x_{3n} : n = 1, 2, 3, \ldots\}$ is

$$\sum_0^{p_1+p_2} \sum_{q+r=i} c_{1q} c_{2r} x_{3(n+i)} = 0. \ ■$$

Corollary 3.2: If $K_3 = 2K_1$, the recurrence relation for x_{3n} is

$$\sum_0^{p_1} c_{1i} x_{n+i} = 0.$$

Consider the direct product $K_3 = K_1 \times K_2$. Let K_1 be partitioned into four square matrices.

$$K_1 = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad K_1 \times K_2 = \begin{bmatrix} A_1 \times K_2 & A_2 \times K_2 \\ A_3 \times K_2 & A_4 \times K_2 \end{bmatrix}.$$

Let $Q = yI - K_3$, then

$$Q = \begin{bmatrix} yI - A_1 \times K_2 & -A_2 \times K_2 \\ -A_3 \times K_2 & yI - A_4 \times K_2 \end{bmatrix}.$$

Multiply the top row of Q by $(A_3 \times K_2)(yI - A_1 \times K_2)^{-1}$ and add this to the second row [1], then

$$|Q| = \begin{vmatrix} yI - A_1 \times K_2 & -A_2 \times K_2 \\ 0 & yI - A_4 \times K_2 - (A_3 \times K_2)(yI - A_1 \times K_2)^{-1}(A_2 \times K_2) \end{vmatrix}.$$

If A_1 and A_3 commute, then

$$|Q| = |(yI - A_1 \times K_2)(yI - A_4 \times K_2) - (A_3 \times K_2)(A_2 \times K_2)| = 0$$

is the characteristic equation for K_3. This reduces to

$$|y^2I - y(A_1 + A_4) \times K_2 + (A_1A_4 - A_3A_2) \times K_2^2| = 0. \tag{3.2}$$

The recurrence relation for the sequence $\{x_{3n} : n = 1, 2, 3, \ldots\}$ may then be derived if K_1 and K_2 are known.

Example: Let $K_1 = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ where $K_2 = A_1 = A_2 = A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $A_4 = 0$. From Equation (3.2), the characteristic equation is

$$|yI - K_3| = \left| y^2I - y \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right| = 0$$

or

$$y^8 - y^7 - 13y^6 - 8y^5 + 20y^4 + 8y^3 - 13y^2 + y + 1 = 0.$$

The recurrence relation for the sequence $\{x_n = (F_{n+2})^3\}$ is

$$x_{n+8} - x_{n+7} - 13x_{n+6} - 8x_{n+5} + 20x_{n+4} + 8x_{n+3} - 13x_{n+2} + x_{n+1} + x_n = 0.$$

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