EXPANSION OF THE FIBONACCI NUMBERS F_{mn+r} IN THE m^{th} POWERS OF FIBONACCI OR LUCAS NUMBERS

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1. INTRODUCTION

It is known, see [1, p. 77], that

$$F_{2a}F_{2n} = F_{a+n}^2 - F_{a-n}^2$$

and, see [2, p. 43], that

$$F_a F_{2a} F_{3n} = F_{a+n}^3 + (-1)^{a+1} L_a F_n^3 + (-1)^{n+1} F_{a-n}^3$$

for arbitrary integers a and n. These identities suggest the possible existence of a general identity of the form

$$wF_{mn} = \sum_{t=1}^{k} b_t [F_{ta+n}^m + (-1)^{mn+1} F_{ta-n}^m] + b, \tag{1}$$

where m, n, and a are integers with $m > 0$, and where w and b_t, $1 \leqslant t \leqslant k$, are integral expressions free of the variable n, and b is an integral expression. Gladwin [3] has given existence proofs for some general identities of a similar type. An example of the kind of identity that we shall obtain is

$$\begin{aligned} F_a^2 F_{2a}^2 F_{3a} F_{4a} F_{5a} F_{6a} F_{6n} &= F_a F_{2a} F_{3a} F_{n+3a}^6 + (-1)^{a+1} F_{2a}^2 F_{6a} F_{n+2a}^6 \\ &+ (-1)^a F_a F_{5a} F_{6a} F_{n+a}^6 + (-1)^{a+1} F_a F_{5a} F_{6a} F_{n-a}^6 \\ &+ (-1)^a F_{2a}^2 F_{6a} F_{n-2a}^6 - F_a F_{2a} F_{3a} F_{n-3a}^6 \end{aligned}$$

for arbitrary integers a and n. In the sequel we shall use the following well-known results: for all integers a and n,

$$(L_n \pm \sqrt{5}F_n)^m = 2^{m-1}(L_{mn} \pm \sqrt{5}F_{mn}) \tag{2}$$

where m is a positive integer,

$$F_{-n} = (-1)^{n+1} F_n \quad \text{and} \quad L_{-n} = (-1)^n L_n, \tag{3}$$

$$L_n^2 = 5F_n^2 + (-1)^n 4, \tag{4}$$

$$2F_{a+n} = F_a L_n + L_a F_n, \tag{5}$$

$$2L_{a+n} = 5F_a F_n + L_a L_n. \tag{6}$$

2. PRELIMINARY LEMMAS

Lemma 1: $L_m^2/F_m^2 - L_n^2/F_n^2 = (-1)^{n+1}4F_{m+n}F_{m-n}/F_m^2F_n^2$ for $m \neq 0$ and $n \neq 0$.

The proof of Lemma 1 follows from equations (3) and (5).

In the sequel, let a be a nonzero integer.

Lemma 2: For $m > 0$,

$$\text{(i)} \quad 2^{m-1}F_{mn} = \sum_{i=1}^{\left[\frac{m+1}{2}\right]} \binom{m}{2i-1} 5^{i-1}F_n^{2i-1}L_n^{m+1-2i},$$

$$\text{(ii)} \quad 2^{m-1}L_{mn} = \sum_{i=1}^{\left[\frac{m+2}{2}\right]} \binom{m}{2i-2} 5^{i-1}F_n^{2i-2}L_n^{m+2-2i},$$

$$\text{(iii)} \quad 2^{m-1}[F_{a+n}^m + (-1)^{mn+1}F_{a-n}^m] = \sum_{i=1}^{\left[\frac{m+1}{2}\right]} \binom{m}{2i-1} F_n^{2i-1}L_n^{m+1-2i}F_a^{m+1-2i}L_a^{2i-1},$$

$$\text{(iv)} \quad 2^{m-1}[F_{a+n}^m + (-1)^{mn}F_{a-n}^m] = \sum_{i=1}^{\left[\frac{m+2}{2}\right]} \binom{m}{2i-2} F_n^{2i-2}L_n^{m+2-2i}F_a^{m+2-2i}L_a^{2i-2},$$

$$\text{(v)} \quad 2^{m-1}[L_{a+n}^m + (-1)^{mn}L_{a-n}^m] = \sum_{i=1}^{\left[\frac{m+2}{2}\right]} \binom{m}{2i-2} 5^{2i-2}F_n^{2i-2}L_n^{m+2-2i}F_a^{2i-2}L_a^{m+2-2i},$$

$$\text{(vi)} \quad 2^{m-1}[L_{a+n}^m + (-1)^{mn+1}L_{a-n}^m] = \sum_{i=1}^{\left[\frac{m+1}{2}\right]} \binom{m}{2i-1} 5^{2i-1}F_n^{2i-1}L_n^{m+1-2i}F_a^{2i-1}L_a^{m+1-2i}.$$

Proof: We shall prove formulas (i) and (iii). The remaining formulas have similar proofs. By equation (2),

$$2^{m-1}(L_{mn} + \sqrt{5}F_{mn}) - 2^{m-1}(L_{mn} - \sqrt{5}F_{mn}) = (L_n + \sqrt{5}F_n)^m - (L_n - \sqrt{5}F_n)^m.$$

That is,

$$\begin{aligned} 2^m\sqrt{5}F_{mn} &= \sum_{i=1}^{m} \binom{m}{i} L_n^{m-i}(\sqrt{5}F_n)^i[1 + (-1)^{i+1}] \\ &= 2\sum_{i=1,\ i \text{ odd}}^{2\left[\frac{m+1}{2}\right]-1} \binom{m}{i} L_n^{m-i}(\sqrt{5}F_n)^i \\ &= 2\sum_{i=1}^{\left[\frac{m+1}{2}\right]} \binom{m}{2i-1} L_n^{m-2i+1}(\sqrt{5}F_n)^{2i-1}. \end{aligned}$$

Formula (i) can now be obtained by dividing through by $2\sqrt{5}$.

Now, by equations (5) and (3),

$$
\begin{aligned}
2^m[F^m_{a+n} + (-1)^{mn+1}F^m_{a-n}] &= (F_aL_n + L_aF_n)^m + (-1)^{mn+1}(F_aL_{-n} + L_aF_{-n})^m \\
&= (F_aL_n + L_aF_n)^m - (F_aL_n - L_aF_n)^m \\
&= \sum_{i=1}^{m}\binom{m}{i}F_a^{m-i}L_n^{m-i}L_a^iF_n^i[1 + (-1)^{i+1}] \\
&= 2\sum_{i=1,\, i \text{ odd}}^{2\left[\frac{m+1}{2}\right]-1}\binom{m}{i}F_n^iL_n^{m-i}F_a^{m-i}L_a^i \\
&= 2\sum_{i=1}^{\left[\frac{m+1}{2}\right]}\binom{m}{2i-1}F_n^{2i-1}L_n^{m-2i+1}F_a^{m-2i+1}L_a^{2i-1}.
\end{aligned}
$$

Formula (iii) is obtained by dividing through by 2.

Let $V_k = (x_t^{i-1})$ for $1 \leqslant i,\ t \leqslant k$, denote the Vandermonde matrix. From [4, pp. 15, 16] it follows that for $k > 1$ and $t = 1, 2, \ldots, k$,

$$|V_k| = (V_k)_{kt}\prod_{\substack{i=1\\ i\neq t}}^{k}(x_t - x_i), \tag{7}$$

where $(V_k)_{kt}$ is the cofactor of x_t^{k-1} in $|V_k|$.

Lemma 3: For $k > 1$ and any constant $c \neq x_t$, $t = 1, 2, \ldots, k$,

$$\sum_{t=1}^{k}(V_k)_{kt}\Big/(c - x_t) = |V_k|\Big/\prod_{i=1}^{k}(c - x_i).$$

Proof: Let $C_k = [c_{it}]$, where $c_{it} = 1$ if $i = t$, $c_{it} = -c$ if $i = t + 1$, and $c_{it} = 0$ otherwise. Then,

$$
\begin{aligned}
|V_k| = |C_k|\cdot|V_k| = |C_kV_k| = \sum_{t=1}^{k}(C_kV_k)_{1t} &= \sum_{t=1}^{k}\left[\prod_{\substack{i=1\\ i\neq t}}^{k}(c - x_i)\right](V_k)_{kt} \\
&= \prod_{i=1}^{k}(c - x_i)\sum_{t=1}^{k}(V_k)_{kt}\Big/(c - x_t).
\end{aligned}
$$

In the sequel, let $x_t \equiv L^2_{ta}/F^2_{ta}$ for $t = 1, 2, \ldots, k$.

Lemma 4: For $k > 1$ and $t = 1, 2, \ldots, k$,

$$(-1)^{ta+1}2^{2k-2}(V_k)_{kt}\prod_{i=k+1}^{k+t}F_{ia} = L_{ta}F_{ta}^{2k-2}|V_k|\prod_{i=k-t+1}^{k}F_{-ia}.$$

Proof: By equation (7) and Lemma 1, for $k > 1$ and $t = 1, 2, \ldots, k$,

$$
\begin{aligned}
|V_k|/(V_k)_{kt} = \prod_{\substack{i=1\\ i\neq t}}^{k}(x_t - x_i) &= \prod_{\substack{i=1\\ i\neq t}}^{k}[(-1)^{ia+1}4F_{ta+ia}F_{ta-ia}/F^2_{ta}F^2_{ia}] \\
&= (-1)^{ta+1}(F^4_{ta}/4F_{2ta})\left[\prod_{i=1}^{k}[(-1)^{ia+1}4F_{ta+ia}/F^2_{ta}F^2_{ia}]\right]\prod_{\substack{i=1\\ i\neq t}}^{k}F_{ta-ia}
\end{aligned}
$$

$$= (-1)^{ta+1}(4^{k-1}F_{ta}^3/L_{ta})\left[\prod_{i=t+1}^{t+k} F_{ia}\right]\left[\prod_{i=1}^{t-1} F_{ta-ia}\right]\left[\prod_{i=t+1}^{k} F_{ta-ia}\right]$$

$$\div F_{ta}^{2k}\left[\prod_{i=1}^{k} F_{-ia}\right]\left[\prod_{i=1}^{k} F_{ia}\right]$$

$$= (-1)^{ta+1}(4^{k-1}/L_{ta})\left[\prod_{i=t}^{t+k} F_{ia}\right]\left[\prod_{i=1}^{t-1} F_{ia}\right]\left[\prod_{i=1}^{k-t} F_{-ia}\right]$$

$$\div F_{ta}^{2k-2}\left[\prod_{i=1}^{k} F_{-ia}\right]\left[\prod_{i=1}^{k} F_{ia}\right]$$

$$= (-1)^{ta+1}2^{2k-2}\left[\prod_{i=k+1}^{k+t} F_{ia}\right] \Bigg/ L_{ta}F_{ta}^{2k-2}\prod_{i=k-t+1}^{k} F_{-ia}$$

since, by equation (5), $F_{2ta} = F_{ta}L_{ta}$.

Lemma 5: For $k > 1$,

$$\sum_{t=1}^{k}(-1)^{ta+1}F_{ta}^2(V_k)_{kt} = (1/2^{2k-2})|V_k|\left[\prod_{i=1}^{k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}.$$

Proof: For $t = 1, 2, \ldots, k$,

$$x_t \equiv L_{ta}^2/F_{ta}^2 = (5F_{ta}^2 + (-1)^{ta}4)/F_{ta}^2 = 5 + (-1)^{ta}4/F_{ta}^2.$$

It follows that $(-1)^{ta+1}F_{ta}^2 = 4/(5 - x_t)$. Therefore, by Lemma 3,

$$\sum_{t=1}^{k}(-1)^{ta+1}F_{ta}^2(V_k)_{kt} = \sum_{t=1}^{k}(4/(5 - x_t))(V_k)_{kt} = 4|V_k| \Bigg/ \prod_{i=1}^{k}(5 - x_i)$$

$$= 4|V_k| \Bigg/ \prod_{i=1}^{k}((-1)^{ia+1}4/F_{ia}^2)$$

$$= (1/4^{k-1})|V_k|\left[\prod_{i=1}^{k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}.$$

Lemma 6: Let u be a positive integer and let z_t be a real number for $t = 1, 2, \ldots, k$.

$$\sum_{t=1}^{k} z_t x_t^i = 5^i \sum_{t=1}^{k} z_t \quad \text{for each } i,\ 1 \leqslant i \leqslant u,$$

if and only if

$$\sum_{t=1}^{k} z_t x_t^i = 5\sum_{t=1}^{k} z_t x_t^{i-1} \quad \text{for each } i,\ 1 \leqslant i \leqslant u.$$

Proof: Let

$$\sum_{t=1}^{k} z_t x_t^i = 5^i \sum_{t=1}^{k} z_t \quad \text{for each } i,\ 1 \leqslant i \leqslant u.$$

Then, for $i = 1$,

$$\sum_{t=1}^{k} z_t x_t = 5\sum_{t=1}^{k} z_t$$

and, for $2 \leqslant i \leqslant u$,

$$\sum_{t=1}^{k} z_t x_t^i = 5 \cdot 5^{i-1} \sum_{t=1}^{k} z_t = 5 \sum_{t=1}^{k} z_t x_t^{i-1}.$$

Conversely, we use mathematical induction on u. The case $u = 1$ is true. For $q \geqslant 1$, assume that

$$\sum_{t=1}^{k} z_t x_t^i = 5 \sum_{t=1}^{k} z_t x_t^{i-1} \quad \text{for each } i,\ 1 \leqslant i \leqslant q,$$

implies

$$\sum_{t=1}^{k} z_t x_t^i = 5^i \sum_{t=1}^{k} z_t \quad \text{for each } i,\ 1 \leqslant i \leqslant q.$$

Now let

$$\sum_{t=1}^{k} z_t x_t^i = 5 \sum_{t=1}^{k} z_t x_t^{i-1} \quad \text{for each } i,\ 1 \leqslant i \leqslant q + 1.$$

Then

$$\sum_{t=1}^{k} z_t x_t^i = 5 \sum_{t=1}^{k} z_t x_t^{i-1} \quad \text{for each } i,\ 1 \leqslant i \leqslant q$$

and

$$\sum_{t=1}^{k} z_t x_t^{q+1} = 5 \sum_{t=1}^{k} z_t x_t^{q}.$$

Therefore, by the induction hypothesis,

$$\sum_{t=1}^{k} z_t x_t^i = 5^i \sum_{t=1}^{k} z_t \quad \text{for each } i,\ 1 \leqslant i \leqslant q$$

and

$$\sum_{t=1}^{k} z_t x_t^{q+1} = 5 \sum_{t=1}^{k} z_t x_t^{q}.$$

Hence

$$\sum_{t=1}^{k} z_t x_t^i = 5^i \sum_{t=1}^{k} z_t \quad \text{for each } i,\ 1 \leqslant i \leqslant q,$$

and

$$\sum_{t=1}^{k} z_t x_t^{q+1} = 5 \cdot 5^q \sum_{t=1}^{k} z_t = 5^{q+1} \sum_{t=1}^{k} z_t.$$

Thus

$$\sum_{t=1}^{k} z_t x_t^i = 5^i \sum_{t=1}^{k} z_t \quad \text{for each } i,\ 1 \leqslant i \leqslant q + 1.$$

The proof is complete by mathematical induction.

Lemma 7: Let z_t be a real number for $t = 1, 2, \ldots, k$, and let j be a fixed integer.

$$\sum_{t=1}^{k} (-1)^{ta} z_t x_t^{j-1}/F_{ta}^2 = 0 \text{ if and only if } \sum_{t=1}^{k} z_t x_t^j = 5 \sum_{t=1}^{k} z_t x_t^{j-1}.$$

Proof:

$$\begin{aligned}\sum_{t=1}^{k} z_t x_t^j &= \sum_{t=1}^{k} z_t (L_{ta}^2/F_{ta}^2) x_t^{j-1} \\ &= \sum_{t=1}^{k} z_t ((5F_{ta}^2 + (-1)^{ta} 4)/F_{ta}^2) x_t^{j-1} \\ &= 5 \sum_{t=1}^{k} z_t x_t^{j-1} + 4 \sum_{t=1}^{k} (-1)^{ta} z_t x_t^{j-1}/F_{ta}^2.\end{aligned}$$

Corollary 1: Let z_t be a real number for $t = 1, 2, \ldots, k$.

$$\sum_{t=1}^{k}(-1)^{ta} z_t x_t^{i-1}/F_{ta}^2 = 0 \quad \text{for each } i,\ 1 \leqslant i \leqslant u$$

if and only if

$$\sum_{t=1}^{k} z_t x_t^i = 5^i \sum_{t=1}^{k} z_t \quad \text{for each } i,\ 1 \leqslant i \leqslant u.$$

Proof: Apply Lemma 6 and Lemma 7.

Corollary 2: Let $k > 1$. Then

$$\sum_{t=1}^{k}(-1)^{ta} F_{ta}^2 x_t^{i-1}(V_k)_{kt} = 5^{i-1}\sum_{t=1}^{k}(-1)^{ta} F_{ta}^2 (V_k)_{kt}$$

for each i, $1 \leqslant i \leqslant k$.

Proof: In Corollary 1, let $z_t = (-1)^{ta} F_{ta}^2 (V_k)_{kt}$ for $t = 1, 2, \ldots, k$, and let $u = k - 1$. Then

$$\sum_{t=1}^{k}(-1)^{ta} z_t x_t^{i-1}/F_{ta}^2 = \sum_{t=1}^{k} x_t^{i-1}(V_k)_{kt} = 0$$

for each i, $1 \leqslant i \leqslant k - 1$, since a determinant with two identical rows has numerical value zero. By Corollary 1,

$$\sum_{t=1}^{k} z_t x_t^i = 5^i \sum_{t=1}^{k} z_t$$

is true for each i, $1 \leqslant i \leqslant k - 1$, and clearly is true for $i = 0$. Therefore,

$$\sum_{t=1}^{k}(-1)^{ta} F_{ta}^2 x_t^{i-1}(V_k)_{kt} = 5^{i-1}\sum_{t=1}^{k}(-1)^{ta} F_{ta}^2 (V_k)_{kt}$$

for each i, $1 \leqslant i \leqslant k$.

3. THEOREMS

Theorem 1: For any positive integer k,

$$\text{(i)}\quad 5^{j-1}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia} = \sum_{t=1}^{k}\left[\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\prod_{i=k-t+1}^{k} F_{-ia}\right] F_{ta}^{2k+2-2j} L_{ta}^{2j-1}$$

for $1 \leqslant j \leqslant k$, and

$$\text{(ii)}\quad 5^{k}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia} = \sum_{t=1}^{k}\left[\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\prod_{i=k-t+1}^{k} F_{-ia}\right] L_{ta}^{2k+1} + 2^{2k}\prod_{i=k+1}^{2k} F_{ia},$$

$$\text{and (iii)}\quad \left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia} = 5\sum_{t=1}^{k}\left[\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\prod_{i=k-t+1}^{k} F_{-ia}\right] F_{ta}^{2k+2} L_{ta}^{-1}$$

$$+ (-1)^k 2^{2k}\left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2.$$

Proof: The three identities are easily verified for $k = 1$. Assume that $k > 1$. Denote

$$A = \left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia} \quad \text{and} \quad A_t = \left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\prod_{i=k-t+1}^{k} F_{-ia}$$

for $t = 1, 2, \ldots, k$, and

$$K = -|V_k| \Big/ 2^{2k-2}\prod_{i=k+1}^{2k} F_{ia}.$$

By lemma 5,

$$KA = \sum_{t=1}^{k} (-1)^{ta} F_{ta}^2 (V_k)_{kt} \tag{8}$$

and, by Lemma 4,

$$KA_t = (-1)^{ta}(V_k)_{kt}/L_{ta}F_{ta}^{2k-2} \tag{9}$$

for $t = 1, 2, \ldots, k$. Now, by Corollary 2 and equations (8) and (9) we have, for each j, $1 \leqslant j \leqslant k$,

$$\begin{aligned} 5^{j-1}KA &= 5^{j-1}\sum_{t=1}^{k}(-1)^{ta}F_{ta}^2(V_k)_{kt} = \sum_{t=1}^{k}(-1)^{ta}F_{ta}^2 x_t^{j-1}(V_k)_{kt} \\ &= \sum_{t=1}^{k} KA_t x_t^{j-1}L_{ta}F_{ta}^{2k} = \sum_{t=1}^{k} KA_t(L_{ta}^{2j-2}/F_{ta}^{2j-2})L_{ta}F_{ta}^{2k}. \end{aligned}$$

Therefore, for each j, $1 \leqslant j \leqslant k$,

$$5^{j-1}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia} = \sum_{t=1}^{k}\left[\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\prod_{i=k-t+1}^{k}F_{-ia}\right]F_{ta}^{2k-2j+2}L_{ta}^{2j-1}. \tag{10}$$

The proof of (i) is complete.

From equation (10), we obtain, for $j = k$,

$$\begin{aligned} 5^{k-1}A &= \sum_{t=1}^{k}A_t F_{ta}^2 L_{ta}^{2k-1} = (1/5)\sum_{t=1}^{k}A_t(L_{ta}^2 + (-1)^{ta+1}4)L_{ta}^{2k-1} \\ &= (1/5)\sum_{t=1}^{k}A_t L_{ta}^{2k+1} - (4/5)\sum_{t=1}^{k}(-1)^{ta}A_t L_{ta}^{2k-1} \\ &= (1/5)\sum_{t=1}^{k}A_t L_{ta}^{2k+1} - (4/5K)\sum_{t=1}^{k}(L_{ta}^{2k-2}/F_{ta}^{2k-2})(V_k)_{kt} \end{aligned}$$

by equation (9). Thus,

$$\begin{aligned} 5^{k-1}A &= (1/5)\sum_{t=1}^{k}A_t L_{ta}^{2k+1} - (4/5K)\sum_{t=1}^{k}x_t^{k-1}(V_k)_{kt} \\ &= (1/5)\sum_{t=1}^{k}A_t L_{ta}^{2k+1} - (4/5K)|V_k| \end{aligned}$$

$$= (1/5)\sum_{t=1}^{k} A_t L_{ta}^{2k+1} + (2^{2k}/5)\prod_{i=k+1}^{2k} F_{ia}.$$

The proof of (ii) is complete.

From equation (10) we obtain, for $j = 1$,

$$A = \sum_{t=1}^{k} A_t F_{ta}^{2k} L_{ta} = \sum_{t=1}^{k} A_t F_{ta}^{2k}(5F_{ta}^2 + (-1)^{ta}4)/L_{ta}$$

$$= 5\sum_{t=1}^{k} A_t F_{ta}^{2k+2} L_{ta}^{-1} + 4\sum_{t=1}^{k} A_t (-1)^{ta} F_{ta}^{2k} L_{ta}^{-1}$$

$$= 5\sum_{t=1}^{k} A_t F_{ta}^{2k+2} L_{ta}^{-1} + (4/K)\sum_{t=1}^{k} (F_{ta}^2/L_{ta}^2)(V_k)_{kt}$$

$$= 5\sum_{t=1}^{k} A_t F_{ta}^{2k+2} L_{ta}^{-1} + (4/K)\sum_{t=1}^{k} (1/x_t)(V_k)_{kt}.$$

Therefore, by Lemma 3,

$$A = 5\sum_{t=1}^{k} A_t F_{ta}^{2k+2} L_{ta}^{-1} + (4/K)(-1)|V_k| \Big/ \prod_{i=1}^{k}(-x_i)$$

$$= 5\sum_{t=1}^{k} A_t F_{ta}^{2k+2} L_{ta}^{-1} + (-1)^k 2^{2k}\left[\prod_{i=k+1}^{2k} F_{ia}\right]\left[\prod_{i=1}^{k} F_{ia}^2\right] \Big/ \prod_{i=1}^{k} L_{ia}^2$$

$$= 5\sum_{t=1}^{k} A_t F_{ta}^{2k+2} L_{ta}^{-1} + (-1)^k 2^{2k}\left[\prod_{i=1}^{2k} F_{ia}\right]\left[\prod_{i=1}^{k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2.$$

The proof of (iii) is complete.

Lemma 8: Let a and n be nonzero integers, let k and m be positive integers, and let $\varepsilon = 0$ or $\varepsilon = 1$.

(i) For $m \leqslant 2k + 2 + 2\varepsilon$, $2^{m-1}F_{mn}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$

$$= 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k} F_{ta}^{2k+1-m+2\varepsilon} L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right]\left[F_{ta+n}^m + (-1)^{mn+1}F_{ta-n}^m\right]$$

$$+ (-1)^k\binom{m}{2\varepsilon - 1}2^{2k}F_n L_n^{m-1}\left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \binom{m}{2k + 1 + 2\varepsilon}5^{\varepsilon}2^{2k}F_n^{2k+1+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k} F_{ia},$$

(ii) For $m \leqslant 2k + 2 + 2\varepsilon$, $5^{k+\varepsilon}2^{m-1}F_{mn}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$

$$= 2^{m-1}\sum_{t=1}^{k} F_{ta}^{1-2\varepsilon} L_{ta}^{2k-m+2\varepsilon}\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right]\left[L_{ta+n}^m + (-1)^{mn+1}L_{ta-n}^m\right]$$

$$+ \binom{m}{2\varepsilon - 1}5 \cdot 2^{2k}F_n L_n^{m-1}\prod_{i=k+1}^{2k} F_{ia} +$$

(continued)

$$+ (-1)^k \binom{m}{2k+1+2\varepsilon} 5^{2k+2\varepsilon} 2^{2k} F_n^{2k+1+2\varepsilon} L_n^{m-2k-1-2\varepsilon} \left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2,$$

(iii) For $m \leqslant 2k + 1 + 2\varepsilon$, $5^{k-1+\varepsilon} 2^{m-1} L_{mn} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$

$$= 2^{m-1} \sum_{t=1}^{k} F_{ta}^{2-2\varepsilon} L_{ta}^{2k-m-1+2\varepsilon} \left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right]\left[L_{ta+n}^m + (-1)^{mn} L_{ta-n}^m\right]$$

$$+ \binom{m}{2\varepsilon - 2} 2^{2k} L_n^m \prod_{i=k+1}^{2k} F_{ia}$$

$$+ (-1)^k \binom{m}{2k+2\varepsilon} 5^{2k-1+2\varepsilon} 2^{2k} F_n^{2k+2\varepsilon} L_n^{m-2k-2\varepsilon} \left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2,$$

(iv) For $m \leqslant 2k + 1 + 2\varepsilon$, $2^{m-1} L_{mn} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$

$$= 5^{\varepsilon} \cdot 2^{m-1} \sum_{t=1}^{k} F_{ta}^{2k-m+2\varepsilon} L_{ta}^{1-2\varepsilon} \left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right]\left[F_{ta+n}^m + (-1)^{mn} F_{ta-n}^m\right]$$

$$+ (-1)^k \binom{m}{2\varepsilon - 2} 2^{2k} L_n^m \left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \binom{m}{2k+2\varepsilon} 5^{\varepsilon} 2^{2k} F_n^{2k+2\varepsilon} L_n^{m-2k-2\varepsilon} \prod_{i=k+1}^{2k} F_{ia}.$$

Proof: (i) Let $F_{ta}^{2k+1-m+2\varepsilon} L_{ta}^{-2\varepsilon} \left[\prod_{i=k+t+1}^{2k} F_{ia}\right] \prod_{i=k-t+1}^{k} F_{-ia} = b_t$ for $1 \leqslant t \leqslant k$. Then, by Theorem 1 (i),

$$5^{j-1} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia} = \sum_{t=1}^{k} b_t F_{ta}^{m+1-2j-2\varepsilon} L_{ta}^{2j-1+2\varepsilon} \quad \text{for } 1 \leqslant j \leqslant k.$$

Thus,

$$5^{j-1-\varepsilon} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia} = \sum_{t=1}^{k} b_t F_{ta}^{m+1-2j} L_{ta}^{2j-1} \quad \text{for } 1 + \varepsilon \leqslant j \leqslant k + \varepsilon.$$

So

$$\binom{m}{2j-1} F_n^{2j-1} L_n^{m+1-2j} 5^{j-1} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon} \sum_{t=1}^{k} b_t \binom{m}{2j-1} F_n^{2j-1} L_n^{m+1-2j} F_{ta}^{m+1-2j} L_{ta}^{2j-1} \quad \text{for } 1 + \varepsilon \leqslant j \leqslant k + \varepsilon.$$

Since, by hypothesis, $m \leqslant 2k + 2 + 2\varepsilon$, we have $[(m-1)/2] \leqslant k + \varepsilon$. Therefore,

$$\sum_{j=1+\varepsilon}^{\left[\frac{m-1}{2}\right]} \binom{m}{2j-1} F_n^{2j-1} L_n^{m+1-2j} 5^{j-1} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon} \sum_{t=1}^{k} b_t \sum_{j=1+\varepsilon}^{\left[\frac{m-1}{2}\right]} \binom{m}{2j-1} F_n^{2j-1} L_n^{m+1-2j} F_{ta}^{m+1-2j} L_{ta}^{2j-1}.$$

By Theorem 1 (ii),

$$5^k\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia} = \sum_{t=1}^{k} b_t F_{ta}^{m-2k-1-2\varepsilon} L_{ta}^{2k+1+2\varepsilon} + 2^{2k}\prod_{i=k+1}^{2k} F_{ia},$$

and by Theorem 1 (iii),

$$5^{-1}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$$

$$= \sum_{t=1}^{k} b_t F_{ta}^{m+1-2\varepsilon} L_{ta}^{2\varepsilon-1} + (-1)^k 2^{2k}\left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ 5\prod_{i=1}^{k} L_{ia}^2.$$

Therefore,

$$\binom{m}{2k+1+2\varepsilon} F_n^{2k+1+2\varepsilon} L_n^{m-2k-1-2\varepsilon} 5^{k+\varepsilon}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon}\sum_{t=1}^{k} b_t \binom{m}{2k+1+2\varepsilon} F_n^{2k+1+2\varepsilon} L_n^{m-2k-1-2\varepsilon} F_{ta}^{m-2k-1-2\varepsilon} L_{ta}^{2k+1+2\varepsilon}$$

$$+ \binom{m}{2k+1+2\varepsilon} 5^{\varepsilon} 2^{2k} F_n^{2k+1+2\varepsilon} L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k} F_{ia}$$

and

$$\binom{m}{2\varepsilon-1} F_n^{2\varepsilon-1} L_n^{m+1-2\varepsilon} 5^{\varepsilon-1}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon}\sum_{t=1}^{k} b_t \binom{m}{2\varepsilon-1} F_n^{2\varepsilon-1} L_n^{m+1-2\varepsilon} F_{ta}^{m+1-2\varepsilon} L_{ta}^{2\varepsilon-1}$$

$$+ (-1)^k\binom{m}{2\varepsilon-1} 5^{\varepsilon} 2^{2k} F_n^{2\varepsilon-1} L_n^{m+1-2\varepsilon}\left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ 5\prod_{i=1}^{k} L_{ia}^2.$$

Since, by hypothesis, $m \leqslant 2k + 2 + 2\varepsilon$, we have $[(m+1)/2] \leqslant k + 1 + \varepsilon$, and we have $\binom{m}{2k+1+2\varepsilon} = 0$ if and only if $[(m+1)/2] < k + 1 + \varepsilon$. Therefore,

$$\binom{m}{2k+1+2\varepsilon} F_n^{2[(m+1)/2]-1} L_n^{m+1-2[(m+1)/2]} 5^{[(m-1)/2]}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon}\sum_{t=1}^{k} b_t \binom{m}{2k+1+2\varepsilon} (F_n L_{ta})^{2[(m+1)/2]-1} (L_n F_{ta})^{m+1-2[(m+1)/2]}$$

$$+ \binom{m}{2k+1+2\varepsilon} 5^{\varepsilon} 2^{2k} F_n^{2k+1+2\varepsilon} L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k} F_{ia}.$$

Since $\binom{m}{-1} = 0$, we have

$$\sum_{j=1}^{\left[\frac{m+1}{2}\right]} \binom{m}{2j-1} F_n^{2j-1} L_n^{m+1-2j} 5^{j-1}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon}\sum_{t=1}^{k} b_t \sum_{j=1}^{\left[\frac{m+1}{2}\right]} \binom{m}{2j-1} F_n^{2j-1} L_n^{m+1-2j} F_{ta}^{m+1-2j} L_{ta}^{2j-1} +$$

(continued)

$$+ (-1)^k\binom{m}{2\varepsilon - 1}2^{2k}F_nL_n^{m-1}\left[\prod_{i=1}^{k}F_{ia}\right]\left[\prod_{i=1}^{2k}F_{ia}\right]\Big/\prod_{i=1}^{k}L_{ia}^2$$

$$+ \binom{m}{2k + 1 + 2\varepsilon}5^{\varepsilon}2^{2k}F_n^{2k+1+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k}F_{ia}.$$

By Lemma 2 (i) and Lemma 2 (iii),

$$2^{m-1}F_{mn}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$$

$$= 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k}b_t[F_{ta+n}^m + (-1)^{mn+1}F_{ta-n}^m]$$

$$+ (-1)^k\binom{m}{2\varepsilon - 1}2^{2k}F_nL_n^{m-1}\left[\prod_{i=1}^{k}F_{ia}\right]\left[\prod_{i=1}^{2k}F_{ia}\right]\Big/\prod_{i=1}^{k}L_{ia}^2$$

$$+ \binom{m}{2k + 1 + 2\varepsilon}5^{\varepsilon}2^{2k}F_n^{2k+1+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k}F_{ia}.$$

After substitution for b_t, $1 \leqslant t \leqslant k$, the proof of (i) is complete. The proofs of (ii), (iii), and (iv) are similar.

From equations (5) and (6), we obtain the following four identities:

$$L_n + F_n = 2F_{n+1},\quad L_n - F_n = 2F_{n-1},\quad 5F_n + L_n = 2L_{n+1},\quad 5F_n - L_n = 2L_{n-1}$$

for all integers n.

Corollary 3: Let a and n be nonzero integers and let k and m be positive integers and let $\varepsilon = 0$ or $\varepsilon = 1$. For $m \leqslant 2k + 1 + 2\varepsilon$,

(i) $$2^{m-1}F_{mn+1}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$$

$$= 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k}F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\left[\prod_{i=k-t+1}^{k}F_{-ia}\right]\left[F_{ta+1}F_{ta+n}^m + (-1)^{mn}F_{ta-1}F_{ta-n}^m\right]$$

$$+ (-1)^k\left[\binom{m}{2\varepsilon - 2}L_n + \binom{m}{2\varepsilon - 1}F_n\right]2^{2k-1}L_n^{m-1}\left[\prod_{i=1}^{k}F_{ia}\right]\left[\prod_{i=1}^{2k}F_{ia}\right]\Big/\prod_{i=1}^{k}L_{ia}^2$$

$$+ \left[\binom{m}{2k + 2\varepsilon}L_n + \binom{m}{2k + 1 + 2\varepsilon}F_n\right]5^{\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k}F_{ia},$$

and

(ii) $$2^{m-1}L_{mn+1}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$$

$$= 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k}F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\left[\prod_{i=k-t+1}^{k}F_{-ia}\right]\left[L_{ta+1}F_{ta+n}^m + (-1)^{mn+1}L_{ta-1}F_{ta-n}^m\right]$$

$$+ (-1)^k\left[\binom{m}{2\varepsilon - 2}L_n + \binom{m}{2\varepsilon - 1}5F_n\right]2^{2k-1}L_n^{m-1}\left[\prod_{i=1}^{k}F_{ia}\right]\left[\prod_{i=1}^{2k}F_{ia}\right]\Big/\prod_{i=1}^{k}L_{ia}^2$$

$$+ \left[\binom{m}{2k + 2\varepsilon}L_n + \binom{m}{2k + 1 + 2\varepsilon}5F_n\right]5^{\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k}F_{ia},$$

and

(iii) $5^{k+\varepsilon}2^{m-1}F_{mn+1}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$

$$= 2^{m-1}\sum_{t=1}^{k}F_{ta}^{1-2\varepsilon}L_{ta}^{2k-m-1+2\varepsilon}\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\left[\prod_{i=k-t+1}^{k}F_{-ia}\right]\left[L_{ta+1}L_{ta+n}^{m}+(-1)^{mn}L_{ta-1}L_{ta-n}^{m}\right]$$

$$+\left[\binom{m}{2\varepsilon-2}L_n+\binom{m}{2\varepsilon-1}F_n\right]5\cdot 2^{2k-1}L_n^{m-1}\prod_{i=k+1}^{2k}F_{ia}+(-1)^k\left[\binom{m}{2k+2\varepsilon}L_n\right.$$

$$\left.+\binom{m}{2k+1+2\varepsilon}F_n\right]5^{2k+2\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\left[\prod_{i=1}^{k}F_{ia}\right]\left[\prod_{i=1}^{2k}F_{ia}\right]\Big/\prod_{i=1}^{k}L_{ia}^2,$$

and

(iv) $5^{k-1+\varepsilon}2^{m-1}L_{mn+1}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$

$$= 2^{m-1}\sum_{t=1}^{k}F_{ta}^{1-2\varepsilon}L_{ta}^{2k-m-1+2\varepsilon}\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\left[\prod_{i=k-t+1}^{k}F_{-ia}\right]\left[F_{ta+1}L_{ta+n}^{m}+(-1)^{mn+1}F_{ta-1}L_{ta-n}^{m}\right]$$

$$+\left[\binom{m}{2\varepsilon-2}L_n+\binom{m}{2\varepsilon-1}5F_n\right]2^{2k-1}L_n^{m-1}\prod_{i=k+1}^{2k}F_{ia}+(-1)^k\left[\binom{m}{2k+2\varepsilon}L_n\right.$$

$$\left.+\binom{m}{2k+1+2\varepsilon}5F_n\right]5^{2k-1+2\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\left[\prod_{i=1}^{k}F_{ia}\right]\left[\prod_{i=1}^{2k}F_{ia}\right]\Big/\prod_{i=1}^{k}L_{ia}^2.$$

Proof: (1) and (ii) follow from Lemma 8, parts (i) and (iv). (iii) and (iv) follow from Lemma 8, parts (ii) and (iii).

Theorem 2: Let a and n be nonzero integers, let k and m be positive integers, let $\varepsilon = 0$ or $\varepsilon = 1$, and let r be an integer. For $m \leqslant 2k + 1 + 2\varepsilon$,

(i) $2^{m-1}F_{mn+r}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$

$$= 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k}F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\left[\prod_{i=k-t+1}^{k}F_{-ia}\right]$$

$$\times\left[F_{ta+r}F_{ta+n}^{m}+(-1)^{mn+1+r}F_{ta-r}F_{ta-n}^{m}\right]$$

$$+(-1)^k\left[\binom{m}{2\varepsilon-2}F_rL_n+\binom{m}{2\varepsilon-1}L_rF_n\right]2^{2k-1}L_n^{m-1}\left[\prod_{i=1}^{k}F_{ia}\right]\left[\prod_{i=1}^{2k}F_{ia}\right]\Big/\prod_{i=1}^{k}L_{ia}^2$$

$$+\left[\binom{m}{2k+2\varepsilon}F_rL_n+\binom{m}{2k+1+2\varepsilon}L_rF_n\right]5^{\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k}F_{ia},$$

and

(ii) $2^{m-1}L_{mn+r}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$

$$= 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k}F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\left[\prod_{i=k-t+1}^{k}F_{-ia}\right]\left[L_{ta+r}F_{ta+n}^{m}+(-1)^{mn+r}L_{ta-r}F_{ta-n}^{m}\right]$$

$$+ (-1)^k \left[\binom{m}{2\varepsilon - 2} L_r L_n + \binom{m}{2\varepsilon - 1} 5F_r F_n\right] 2^{2k-1} L_n^{m-1} \left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \left[\binom{m}{2k + 2\varepsilon} L_r L_n + \binom{m}{2k + 1 + 2\varepsilon} 5F_r F_n\right] 5^{\varepsilon} 2^{2k-1} F_n^{2k+2\varepsilon} L_n^{m-2k-1-2\varepsilon} \prod_{i=k+1}^{2k} F_{ia},$$

and

(iii) $5^{k+\varepsilon} 2^{m-1} F_{mn+r} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$

$$= 2^{m-1} \sum_{t=1}^{k} F_{ta}^{1-2\varepsilon} L_{ta}^{2k-m-1+2\varepsilon} \left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right] \times$$

$$\times \left[L_{ta+r} L_{ta+n}^m + (-1)^{mn+1+r} L_{ta-r} L_{ta-n}^m\right]$$

$$+ \left[\binom{m}{2\varepsilon - 2} F_r L_n + \binom{m}{2\varepsilon - 1} L_r F_n\right] 5 \cdot 2^{2k-1} L_n^{m-1} \prod_{i=k+1}^{2k} F_{ia} + (-1)^k \left[\binom{m}{2k + 2\varepsilon} F_r L_n\right.$$

$$\left. + \binom{m}{2k + 1 + 2\varepsilon} L_r F_n\right] 5^{2k+2\varepsilon} 2^{2k-1} F_n^{2k+2\varepsilon} L_n^{m-2k-1-2\varepsilon} \left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2,$$

and

(iv) $5^{k-1+\varepsilon} 2^{m-1} L_{mn+r} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$

$$= 2^{m-1} \sum_{t=1}^{k} F_{ta}^{1-2\varepsilon} L_{ta}^{2k-m-1+2\varepsilon} \left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right]\left[F_{ta+r} L_{ta+n}^m + (-1)^{mn+r} F_{ta-r} L_{ta-n}^m\right]$$

$$+ \left[\binom{m}{2\varepsilon - 2} L_r L_n + \binom{m}{2\varepsilon - 1} 5F_r F_n\right] 2^{2k-1} L_n^{m-1} \prod_{i=k+1}^{2k} F_{ia} + (-1)^k \left[\binom{m}{2k + 2\varepsilon} L_r L_n\right.$$

$$\left. + \binom{m}{2k + 1 + 2\varepsilon} 5F_r F_n\right] 5^{2k-1+2\varepsilon} 2^{2k-1} F_n^{2k+2\varepsilon} L_n^{m-2k-1-2\varepsilon} \left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2.$$

Proof: To prove (i), we use mathematical induction on r. The cases $r = 0$ and $r = 1$ are true by Lemma 8 (i) and Corollary 3 (i). Assume that the hypothesis is true for $r = q$ and for $r = q + 1$, where q is an integer. Then

$$2^{m-1} F_{mn+q+2} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$$

$$= 2^{m-1} F_{mn+q+1} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia} + 2^{m-1} F_{mn+q} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon} 2^{m-1} \sum_{t=1}^{k} F_{ta}^{2k-m+2\varepsilon} L_{ta}^{-2\varepsilon} \left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right] (F_{ta+q+1} + F_{ta+q}) F_{ta+n}^m$$

$$+ 5^{\varepsilon} 2^{m-1} \sum_{t=1}^{k} F_{ta}^{2k-m+2\varepsilon} L_{ta}^{-2\varepsilon} \left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right]$$

$$\times (-1)^{mn+q+1} (-F_{ta-q-1} + F_{ta-q}) F_{ta-n}^m + (-1)^k \left[\binom{m}{2\varepsilon - 2} (F_{q+1} + F_q) L_n + \right.$$

$$+ \binom{m}{2\varepsilon - 1}(L_{q+1} + L_q)F_n\Bigg] 2^{2k-1}L_n^{m-1}\left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Bigg/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \left[\binom{m}{2k + 2\varepsilon}(F_{q+1} + F_q)L_n\right.$$

$$\left. + \binom{m}{2k + 1 + 2\varepsilon}(L_{q+1} + L_q)F_n\right] 5^{\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k} F_{ia}$$

by the induction hypothesis. Therefore,

$$2^{m-1}F_{mn+q+2}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k} F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right] F_{ta+q+2}F_{ta+n}^{m}$$

$$+ 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k} F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right] (-1)^{mn+1+q}F_{ta-q-2}F_{ta-n}^{m}$$

$$+ (-1)^k\left[\binom{m}{2\varepsilon - 2}F_{q+2}L_n + \binom{m}{2\varepsilon - 1}L_{q+2}F_n\right] 2^{2k-1}L_n^{m-1}\left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Bigg/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \left[\binom{m}{2k + 2\varepsilon}F_{q+2}L_n + \binom{m}{2k + 1 + 2\varepsilon}L_{q+2}F_n\right] 5^{\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k} F_{ia}.$$

Similarly,

$$2^{m-1}F_{mn+q-1}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k} F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right] F_{ta+q-1}F_{ta+n}^{m}$$

$$+ 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k} F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right] (-1)^{mn+q}F_{ta-q+1}F_{ta-n}^{m}$$

$$+ (-1)^k\left[\binom{m}{2\varepsilon - 2}F_{q-1}L_n + \binom{m}{2\varepsilon - 1}L_{q-1}F_n\right] 2^{2k-1}L_n^{m-1}\left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Bigg/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \left[\binom{m}{2k + 2\varepsilon}F_{q-1}L_n + \binom{m}{2k + 1 + 2\varepsilon}L_{q-1}F_n\right] 5^{\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k} F_{ia}.$$

The proof of (i) is complete by mathematical induction. The proofs of (ii), (iii), and (iv) are similar.

The three identities given as examples in the introduction can be obtained as special cases of Theorem 2 (i) by using the ordered 6-tuple $(\varepsilon, k, m, n, a, r)$ in the forms $(0,1,2,n,a,0)$, $(0,1,3,n,a,0)$, and $(0,3,6,n,a,0)$, respectively. A special case of Theorem 2 (ii) with the ordered 6-tuple $(0,1,3,n,a,0)$ can be found in [6].

The author thanks the referee for the type of proof used in Lemma 3 and for reference number [4] and for suggestions which led to major simplifications of

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