ELEMENTARY PROBLEMS AND SOLUTIONS

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DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-574 *Proposed by Valentina Bakinova, Rondout Valley, NY*

Let $a_1, a_2, \ldots$ be defined by $a_1 = 1$ and $a_{n+1} = [\sqrt{s_n}]$, where $s_n = a_1 + a_2 + \cdots + a_n$ and $[x]$ is the integer with $x - 1 < [x] \leqslant x$. Find a_{100}, s_{100}, a_{1000}, and s_{1000}.

B-575 *Proposed by L. A. G. Dresel, Reading, England*

Let R_n and S_n be sequences defined by given values R_0, R_1, S_0, S_1 and the recurrence relations $R_{n+1} = rR_n + tR_{n-1}$ and $S_{n+1} = sS_n + tS_{n-1}$, where r, s, t are constants and $n = 1, 2, 3, \ldots$. Show that

$$(r + s)\sum_{k=1}^{n} R_k S_k t^{n-k} = (R_{n+1}S_n + R_n S_{n+1}) - t^n(R_1 S_0 + R_0 S_1).$$

B-576 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $A = L_{2m+3(4n+1)} + (-1)^m$. Show that A is a product of three Fibonacci numbers for all positive integers m and n.

B-577 *Proposed by Herta T. Freitag, Roanoke, VA*

Let A be as in B-575. Show that $4A/5$ is a difference of squares of Fibonacci numbers.

B-578 *Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy*

It is known (Zeckendorf's theorem) that every positive integer N can be represented as a finite sum of distinct nonconsecutive Fibonacci numbers and that this representation is unique. Let $\alpha = (1 + \sqrt{5})/2$ and $[x]$ denote the greatest integer not exceeding x. Denote by $f(N)$ the number of F-addends in the Zeckendorf representation for N. For positive integers n, prove that $f([\alpha F_n]) = 1$ if n is odd.

B-579 *Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy*

Using the notation of B-578, prove that $f([\alpha F_n]) = n/2$ when n is even.

SOLUTIONS

A Specific Fibonacci-Like Sequence

B-550 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Show that the powers of -13 form a Fibonacci-like sequence modulo 181, that is, show that

$$(-13)^{n+1} \equiv (-13)^n + (-13)^{n-1} \pmod{181} \text{ for } n = 1, 2, 3, \ldots .$$

Solution by L. A. G. Dresel, University of Reading, England

We have

$$(-13)^2 = 169 \equiv -13 + 1 \pmod{181},$$

and multiplying by $(-13)^{n-1}$ we obtain

$$(-13)^{n+1} \equiv (-13)^n + (-13)^{n-1} \pmod{181} \text{ for } n = 1, 2, 3, \ldots .$$

Also solved by Paul S. Bruckman, Herta T. Freitag, C. Georghiou, Hans Kappus, L. Kuipers, Bob Prielipp, Helmut Prodinger, Heinz-Jürgen Seiffert, Sahib Singh, Lawrance Somer, J. Suck, Tad White, and the proposer.

A Generalization

B-551 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Generalize on Problem B-550.

Solution by Lawrence Somer, George Washington University, Washington, D.C.

A generalization would be: Let p be an odd prime. Let a and b be integers. Let x be a nonzero residue modulo p. Then

$$x^{n+1} \equiv ax^n + bx^{n-1} \pmod{p} \text{ for } n = 1, 2, 3, \ldots,$$

if and only if $x \equiv (a \pm \sqrt{a^2 + 4b})/2 \pmod{p}$, where $\sqrt{a^2 + 4b}$ is the least positive residue r such that $r^2 \equiv a^2 + 4b \pmod{p}$ if such a residue exists. This result is proved in [1].

Reference

1. L. Somer. "The Fibonacci Group and a New Proof that $F_{p-(5/p)} \equiv 0 \pmod{p}$." *The Fibonacci Quarterly* **10**, no. 4 (1972):345-348, 354.

Also solved by Paul S. Bruckman, L. A. G. Dresel, Herta T. Freitag, C. Georghiou, Hans Kappus, L. Kuipers, Bob Prielipp, Helmut Prodinger, Heinz-Jürgen Seiffert, Sahib Singh, J. Suck, Tad White, and the proposer.

Permutations of 9876543210 Divisible by 11

B-552 *Proposed by Philip L. Mana, Albuquerque, NM*

Let S be the set of integers n with $10^9 < n < 10^{10}$ and with each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 appearing (exactly once) in n.

(a) What is the smallest integer n is S with $11|n$?
(b) What is the probability that $11|n$ for a randomly chosen n in S?

Solution by L. A. G. Dresel, University of Reading, England

Let us number the digit positions 1 to 10 from left to right, and let P_1 denote the set of odd-numbered positions and P_2 the set of even-numbered positions. For a given $n \in S$, let Q_i be the set of digits occupying P_i, and let q_i be the sum of these digits, for $i = 1, 2$. Since each of the digits 0 to 9 appears exactly once in n, we have $q_1 + q_2 = 45$. But, for divisivility by 11, we require $q_1 \equiv q_2 \pmod{11}$, and therefore we must have $q_1 = 17$ *or* $q_1 = 28$.

(a) Let us assume that the first three digits of the smallest integer n in S which is divisible by 11 are 1, 0, 2, in that order. Then Q_1 contains the digits 1 and 2, and we find that $q_1 = 28$ is not achievable; furthermore, $q_1 = 17$ implies that Q_1 contains the digit 3 as well. Hence, the required smallest n is given by $n = 1024375869$.

(b) Let us enumerate all the sets V_k of five distinct digits with a sum equal to 17. There are exactly 11 such sets, namely:

0 1 2 5 9, 0 1 2 6 8, 0 1 3 4 9, 0 1 3 5 8, 0 1 3 6 7, 0 1 4 5 7,

0 2 3 4 8, 0 2 3 5 7, 0 2 4 5 6, 1 2 3 4 7, 1 2 3 5 6.

For each of these sets V_k ($k = 1, 2, \ldots, 11$), the remaining digits form a complementary set W_k with a sum equal to 28. In the case in which V_k contains the digit 0, there are $4 \times 4!$ ways of placing the digits of V_k in P_1, and $5!$ ways of placing the digits of W_k in P_2, giving in all $4 \times 4! \times 5!$ different numbers of the form (V_k, W_k); but there are also $5!$ ways of placing W_k in P_1, with $5!$ ways of placing V_k in P_2, giving a further $5! \times 5!$ numbers of the form (V_k, W_k). Therefore, the total number of permutations of a particular pair V_k, W_k is $9 \times 4! \times 5!$, and we obtain the same result if the digit 0 is contained in W_k instead of V_k. Now, the total number of integers in S is given by $9 \times 9!$, and of these we have $11 \times 9 \times 4! \times 5!$ divisible by 11. Hence, the probability that $11|n$ is $11 \times 4! \times 5!/(9!)$, which simplifies to 11/126, and is slightly less than 1 in 11.

Also solved by Paul S. Bruckman, L. Kuipers, J. Suck, Tad White, and the proposer.

Lucas Summation

B-553 *Proposed by D. L. Muench, St. John Fisher College, Rochester, NY*

Find a compact form for $\sum_{i=0}^{2n} \binom{2n}{i} L_{i+1}^2$.

Solution by C. Georghiou, University of Patras, Greece

We have, for $n > 0$, with the help of the Binet formulas,

$$\sum_{i=0}^{2n} \binom{2n}{i} L_{i+1}^2 = \sum_{i=0}^{2n} \binom{2n}{i} [\alpha^{2i+2} + \beta^{2i+2} - 2(-1)^i]$$

$$= \alpha^2(1+\alpha^2)^{2n} + \beta^2(1+\beta^2)^{2n}$$

$$= \alpha^2(\alpha 5^{1/2})^{2n} + \beta^2(\beta 5^{1/2})^{2n}$$

$$= 5^n L_{2n+2}.$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Piero Filipponi, Herta T. Freitag, Hans Kappus, L. Kuipers, Graham Lord, Bob Prielipp, Helmut Prodinger, Heinz-Jürgen Seiffert, Sahib Singh, J. Suck, Tad White, and the proposer.

Sum of Two Squares

B-554 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

For all n in $Z^+ = \{1, 2, \ldots\}$, prove that there exist x and y in Z^+ such that

$$(F_{4n-1} + 1)(F_{4n+1} + 1) = x^2 + y^2.$$

Solution by Graham Lord, Princeton, NJ

Using the Binet formulas, we have

$$(F_{4n-1} + 1)(F_{4n+1} + 1) = (a^{4n-1} - b^{4n-1} + \sqrt{5})(a^{4n+1} - b^{4n+1} + \sqrt{5})/5$$

$$= \{a^{8n} - 2(ab)^{4n} + b^{8n} + 2 - (a^2 + b^2)(ab)^{4n-1}$$

$$- \sqrt{5}[(1 + a^2)a^{4n-1} - (1 + b^2)b^{4n-1}] + 5\}/5$$

$$= (a^{4n} - b^{4n})^2/5$$

$$+ \{2 + 3 + 5 + \sqrt{5}[a^{4n}(a - b) + b^{4n}(a - b)]\}/5$$

$$= F_{4n}^2 + L_{2n}^2.$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Piero Filipponi, L. Kuipers, Bob Prielipp, Heinz-Jürgen Seiffert, Sahib Singh, J. Suck, Tad White, C. S. Yang & J. F. Wang, and the proposers.

Sum of Three Squares

B-555 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

For all n in Z^+, prove that there exist x, y, and z in Z^+ such that

$$(F_{2n-1} + 4)(F_{2n+5} + 1) = x^2 + y^2 + z^2.$$

Solution by Bob Prielipp, University of Wisconsin, Oshkosh, WI

We shall show that:

(1) $(F_{2n-1} + 4)(F_{2n+5} + 1) = F^2_{2n+2} + F^2_{n+3} + (L_{n+3} - F_{n-2})^2$ if n is even

and

(2) $(F_{2n-1} + 4)(F_{2n+5} + 1) = F^2_{2n+2} + (3F_{n+2})^2 + (F_{n+2} + F_{n+1})^2$ if n is odd.

[The results referred to below (I_{24}, I_{18}, etc.) can be found on pages 56 and 59 of *Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr., Houghton-Mifflin Company, Boston, 1969.]

We begin by establishing the following preliminary results.

Lemma: $F_{2n-1}F_{2n+5} = F^2_{2n+2} + 4$.

Proof: $F_{2n-1}F_{2n+5} = F_{(2n+2)-3}F_{(2n+2)+3} = F^2_{2n+2} + F^2_3$ [by I_{19}] $= F^2_{2n+2} + 4$.

Corollary: $(F_{2n-1} + 4)(F_{2n+5} + 1) = F^2_{2n+2} + 4F_{2n+5} + F_{2n-1} + 8$.

(1) It suffices to prove that

$$4F_{4k+5} + F_{4k-1} + 8 = F^2_{2k+3} + (L_{2k+3} - F_{2k-2})^2.$$

$$\begin{aligned}
F^2_{2k+3} + (L_{2k+3} - F_{2k-2})^2 &= (F^2_{2k+3} + F^2_{2k-2}) - 2L_{2k+3}F_{2k-2} + L^2_{2k+3} \\
&= 5F_{4k+1} - 2(F_{4k+1} - 5) + (L_{4k+6} - 2) \\
&\quad \text{[by } I_{19}, I_{24}, \text{ and } I_{18}, \text{ respectively]} \\
&= 3F_{4k+1} + (F_{4k+6} + 2F_{4k+5}) + 8 \\
&= 3F_{4k+1} + (3F_{4k+5} + F_{4k+4}) + 8 \\
&= 4F_{4k+5} + (3F_{4k+1} - F_{4k+3}) + 8 \\
&= 4F_{4k+5} - (F_{4k+3} - 3F_{4k+1}) + 8 \\
&= 4F_{4k+5} - (F_{4k} - F_{4k+1}) + 8 \\
&= 4F_{4k+5} + F_{4k-1} + 8.
\end{aligned}$$

(2) It suffices to prove that

$$4F_{4k+3} + F_{4k-3} + 8 = (3F_{2k+1})^2 + (F_{2k+1} + L_{2k})^2.$$

$$
\begin{aligned}
(3F_{2k+1})^2 + (F_{2k+1} + L_{2k})^2 &= 2(5F_{2k+1}^2) + 2F_{2k+1}L_{2k} + L_{2k}^2 \\
&= 2(L_{4k+2} + 2) + 2(F_{4k+1} + 1) + (L_{4k} + 2) \\
&\quad [\text{by } I_{17},\ I_{21}, \text{ and } I_{15}, \text{ respectively}] \\
&= 2L_{4k+2} + L_{4k} + 2F_{4k+1} + 8 \\
&= 2(F_{4k+3} + F_{4k+1}) + (F_{4k} + 2F_{4k-1}) \\
&\qquad + 2F_{4k+1} + 8 \\
&= 2F_{4k+3} + 4F_{4k+1} + F_{4k} + 2F_{4k-1} + 8 \\
&= 3F_{4k+3} + 2F_{4k+1} + 2F_{4k-1} + 8 \\
&= 4F_{4k+3} - (F_{4k+2} - F_{4k+1}) + 2F_{4k-1} + 8 \\
&= 4F_{4k+3} - (F_{4k} - F_{4k-1}) + F_{4k-1} + 8 \\
&= 4F_{4k+3} + (F_{4k-1} - F_{4k-2}) + 8 \\
&= 4F_{4k+3} + F_{4k-3} + 8.
\end{aligned}
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Also solved by Paul S. Bruckman, L. A. G. Dresel, Graham Lord, and the proposers.

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