ON THE OCCURRENCES OF FIBONACCI SEQUENCES IN THE COUNTING OF MATCHINGS IN LINEAR POLYGONAL CHAINS

E. J. FARRELL
*The University of the West Indies, St. Augustine, Trinidad*
*(Submitted January 1984)*

## 1. INTRODUCTION

The graphs considered here will be finite and will have no multiple edges. Let $G$ be such a graph. A *matching* in $G$ is a spanning subgraph whose components are nodes and edges only. We define a *k-matching* in $G$ to be a matching with $k$ edges. When the matching consists of edges only, it will be called a *perfect matching*. The number of perfect matchings in $G$ will be denoted by $\gamma(G)$. The total number of matchings in $G$ will be denoted by $\tau(G)$.

The following example illustrates the above definitions.

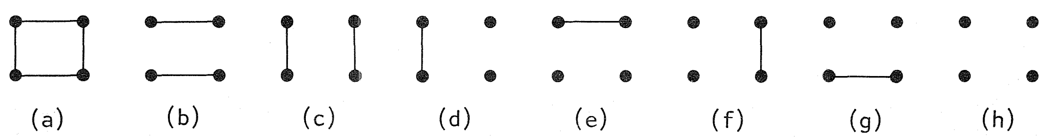



Figure 1

The graph $G$ shown in Figure 1(a) has two perfect matchings [graphs (b) and (c)]. Therefore $\gamma(G) = 2$. $G$ has four 1-matchings [graphs (d), (e), (f), and (g)] and one 0-matching [graph (h)]. Hence $G$ has 7 matchings; i.e., $\tau(G) = 7$.

By a *polygonal chain* $P_{k,n}$, we will mean the graph obtained by concatenating $n$ $k$-gons in such a manner that adjacent $k$-gons (*cells*) have exactly one edge in common. Also, for $k > 3$, no three cells have a common node.

If the first and last cells (cells which are adjacent to exactly one other cell) of $P_{k,n}$ are joined together, so that they have exactly one edge in common, the "circular" structure obtained will be called a *long polygonal chain* $C_{k,n}$. $n$ is called the *length* of the chain. Notice that in $C_{k,n}$, every cell will be adjacent to exactly two cells.

It is clear that different polygonal chains will result, according to the manner in which the cells are concatenated. For example, in the following diagram we show four nonisomorphic versions of $P_{5,4}$.

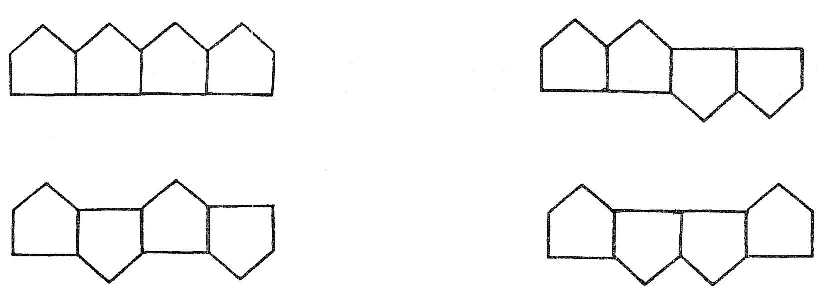


Figure 2

Notice, however, that when $k = 4$ there is only one polygonal chain, $P_{4,n}$. We can also define $P_{4,n}$ as the graph obtained by joining the corresponding nodes of two equal paths with $n$ nodes. We refer to one path as the *upper path* and the other as the *lower path*. The edges in the upper path will be called *upper edges* and those in the lower path will be called *lower edges*.

We define the *linear polygonal chain* $S_{k,n}$ $(k > 3)$ to be the graph obtained from $P_{4,n}$ as follows. If $k$ is even, then every upper and lower edge is replaced by a path of length $(k - 2)/2$. If $k$ is odd, then every upper edge is replaced by a path of length $(k - 1)/2$, and every lower edge is replaced by a path of length $(k - 3)/2$. For $n$-even, $S_{3,n}$ is obtained from $P_{4,(n/2)}$ by joining diagonally opposite nodes in a consistent direction. For $n$-odd, $S_{3,n}$ is obtained from $S_{3,n+1}$ by removing a node of valency 2. The *long linear polygonal chain* $L_{k,n}$ is analogously obtained from $S_{k,n}$, as $C_{k,n}$ is obtained from $P_{k,n}$.

Linear polygonal chains have been the subject of numerous investigations. Their matching polynomials were extensively investigated (see [1, 2, 3, 4, 5]). Polygonal chains have also been called animals, and are special cases of the general animal defined in Harary and Palmer [7]. During investigations of the matching polynomials of linear polygonal chains, it was observed that the number of perfect matchings, and in some cases the total number of matchings, were Fibonacci numbers. These observations form the basis for this report. We refer the reader to Harary [6] for the basic definitions in Graph Theory.

## 2. PRELIMINARY RESULTS

Let $G$ be a graph and $xy$ an edge in $G$ joining nodes $x$ and $y$. We can partition the perfect matchings in $G$ into two classes: (i) those containing $xy$ and (ii) those not containing $xy$. The perfect matchings in class (i) will be perfect matchings in the graph $G''$ obtained from $G$ by removing nodes $x$ and $y$. Those in class (ii) will be perfect matchings in $G'$, the graph obtained from $G$ by deleting the edge $xy$. Thus we have the following lemma.

**Lemma 1:** $\gamma(G) = \gamma(G') + \gamma(G'')$.

Suppose that $G$ consists of two components $H$ and $K$. Then any perfect matchings in $H$ and $K$ can be combined to yield a perfect matching in $G$. Conversely, every perfect matching in $G$ can be broken up into a perfect matching in $H$ and a perfect matching in $K$. Hence we have the following result which generalizes the argument.

**Lemma 2:** Let $G$ be a graph consisting of $r$ components $H_1, H_2, \ldots, H_r$. Then

$$\gamma(G) = \prod_{i=1}^{r} \gamma(H_i).$$

It is clear that if $G$ is a connected graph with an odd number of nodes, then $G$ cannot have a perfect matching.

**Lemma 3:** Let $G$ be a graph. If $G$ has an odd number of nodes, then

$$\gamma(G) = 0.$$

Lemma 1 can be very useful for detecting the polygonal chains $G$ for which $\gamma(G)$ is a Fibonacci number. We simply investigate the relations between $\gamma(G')$ and $\gamma(G'')$ and the chains of shorter lengths. Lemma 2 is useful when applying

Lemma 1, since the deletion of an edge from $G$ might yield a disconnected graph. Lemma 3 is useful for reducing the number of graphs to be considered in applications of Lemma 1.

We can use an argument similar to the one preceding Lemmas 1 and 2 to establish the following analogous results.

**Lemma 4:** $\tau(G) = \tau(G') + \tau(G'')$.

**Lemma 5:** If $G$ consists of $r$ components $H_1, H_2, \ldots, H_r$, then

$$\tau(G) = \prod_{i=1}^{r} \tau(H_i).$$

Lemmas 1 and 4 yield algorithms for counting perfect matchings and total number of matchings, respectively, in graphs. The algorithms consist of repeated applications of the lemmas until graphs $H_i$ are obtained for which $\gamma(H_i)$ and $\tau(H_i)$, respectively, can be written down. These algorithms will be referred to as *reduction processes*. When applying a reduction process, the graph $G'$ will be referred to as the *edge-deleted graph*. $G''$ will be referred to as the *node-deleted graph*.

## 3. THE TRIVIAL CHAINS $S_{1,n}$ AND $L_{1,n}$

We define $P_{1,n}$ to be a tree with nodes of valencies 1 and 2 only. This graph is also called the path or chain $P_n$. When the end-nodes of $P_n$ are identified, the resulting graph $C_{1,n}$ is called the cycle or $n$-gon $C_n$.

Let us apply Lemma 4 to the chain $P_n$ by deleting an edge incident to a node of valency 1. Then $G'$ will contain two components, $P_{n-1}$ and an isolated node $P_1$. Therefore,

$$\tau(G') = \tau(P_{n-1}) \cdot \tau(P_1) = \tau(P_{n-1}).$$

$G''$ will be the graph $P_{n-2}$. Therefore,

$$\tau(G'') = \tau(P_{n-2}).$$

Hence, from Lemma 4, we get

$$\tau(P_n) = \tau(P_{n-1}) + \tau(P_{n-2}).$$

It is clear that $\tau(P_1) = 1$ and $\tau(P_2) = 2$. We define $\tau(P_0) = 1$. Hence we have the following theorem.

**Theorem 1:** The total number of matchings in the chains $P_n$ form a Fibonacci sequence with initial values $\tau(P_0) = \tau(P_1) = 1$.

Let us apply Lemma 4 to the long chain $C_n$. In this case, $G'$ will be the graph $P_n$ and $G''$ will be $P_{n-2}$. Hence we have

$$\tau(C_n) = \tau(P_n) + \tau(P_{n-2}).$$

Therefore,

$$\tau(C_{n-1}) + \tau(C_{n-2}) = \tau(P_{n-1}) + \tau(P_{n-3}) + \tau(P_{n-2}) + \tau(P_{n-4})$$

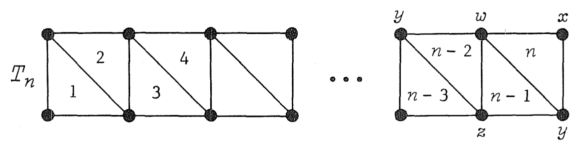
$$= [\tau(P_{n-1}) + \tau(P_{n-2})] + [\tau(P_{n-3}) + \tau(P_{n-4})]$$
$$= \tau(P_n) + \tau(P_{n-2}) = \tau(C_n).$$

Hence we have the following theorem.

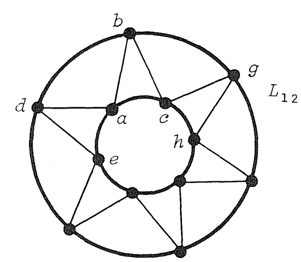
**Theorem 2:** The total number of matchings in the cycles $C_n$ $(n > 2)$ form a Fibonacci sequence with initial values $\tau(C_3) = 4$ and $\tau(C_4) = 7$.

## 4. TRIANGULAR CHAINS

For brevity of notation, we will denote the linear triangular chain $S_{3,n}$ by $T_n$. The long triangular chain $L_{3,n}$ ($n$-even) will be denoted by $L_n$. The graphs $T_n$ and $L_{3,12}$ are shown below in Figures 3(a) and 3(b), respectively.




(a)




(b)

Figure 3

It can be verified that $T_n$ contains $n + 2$ nodes and $2n + 1$ edges. Also $L_n$ contains $n$ nodes and $2n$ edges. Therefore, for odd $n$, $T_n$ and $L_n$ do not have perfect matchings.

Let us apply the reduction process for perfect matchings to the graph $T_n$ ($n$-even) by deleting the edge $xy$ [see Figure 3(a)]. $G'$ will be $T_{n-1}$ with the edge $wx$ attached to it; $G''$ will be $T_{n-2}$. Now, any perfect matching in $G'$ must contain the edge $wx$ since the node $x$ will have valency 1. It follows that the edge $zy$ must also be in every perfect matching of $G'$. The rest of the perfect matching will be a perfect matching of $T_{n-4}$. Hence we get

$$\gamma(G') = \gamma(T_{n-4}).$$

Also,

$$\gamma(G'') = \gamma(T_{n-2}).$$

Therefore, from Lemma 1, we get

$$\gamma(T_n) = \gamma(T_{n-2}) + \gamma(T_{n-4}). \tag{1}$$

It can be confirmed that $\gamma(T_2) = 2$ and $\gamma(T_4) = 3$. We define $\tau(T_0)$ to be 1. Hence we have the following theorem.

**Theorem 3:** The number of perfect matchings in the triangular chains $T_n$ ($n$-even) form a Fibonacci sequence with boundary values $\gamma(T_0) = 1$ and $\gamma(T_2) = 2$.

Let us apply the reduction process for perfect matchings to the graph $L_n$ by deleting the edge $bg$ [see Figure 3(b)]. $G'$ will be $L_n$ with edge $bg$ removed.

$G''$ will be $L_n$ with nodes $b$ and $g$ removed. Let us now apply the reduction process to $G'$ by deleting edge $bc$. Let $G_2'$ be the edge-deleted graph. The graph $G_2''$ obtained by deleting nodes $b$ and $c$ will be $T_{n-4}$.

$$\gamma(G_2'') = \gamma(T_{n-4}).$$

Apply the reduction process to $G_2'$ by deleting edge $ac$. The edge-deleted graph will be $T_{n-2}$. The node-deleted graph will be $T_{n-5}$ with an edge attached to a node of valency 2. Therefore,

$$\gamma(G_2') = \gamma(T_{n-2}) + \gamma(T_{n-6}).$$

Consider now the graph $G''$. We can apply the reduction process by deleting edge $ac$. The edge-deleted graph $G_3'$ will be $T_{n-5}$ with an edge attached to a node of valency 2. Therefore,

$$\gamma(G_3') = \gamma(T_{n-6}).$$

The node-deleted graph will be $T_{n-6}$. Therefore, we get

$$\gamma(G'') = 2\gamma(T_{n-6}).$$

Hence, by adding the contributions of the final graphs, we obtain the following lemma.

**Lemma 6:** $\gamma(L_n) = \gamma(T_{n-2}) + \gamma(T_{n-4}) + 3\gamma(T_{n-6})$ ($n$-even and $n > 4$), with

$$\gamma(T_0) = 1,\ \gamma(T_2) = 2, \text{ and } \gamma(T_4) = 3.$$

The above lemma yields:

$$\begin{aligned}\gamma(L_{n-2}) + \gamma(L_{n-4}) &= \gamma(T_{n-4}) + \gamma(T_{n-6}) + 3\gamma(T_{n-8}) + \gamma(T_{n-6}) \\ &\qquad + \gamma(T_{n-8}) + 3\gamma(T_{n-10}) \\ &= [\gamma(T_{n-4}) + \gamma(T_{n-6})] + [\gamma(T_{n-6}) + \gamma(T_{n-8})] \\ &\qquad + 3[\gamma(T_{n-8}) + \gamma(T_{n-10})] \\ &= \gamma(T_{n-2}) + \gamma(T_{n-4}) + 3\gamma(T_{n-6}), \text{ using Equation (1)} \\ &= \gamma(L_n), \text{ from Lemma 6.}\end{aligned}$$

Thus, we obtain the following result.

**Theorem 4:** The number of perfect matchings in the long triangular chains $L_n$ ($n$-even) form a Fibonacci sequence with initial values $\gamma(L_0) = 4$ (by convention), $\gamma(L_2) = 2$, and $\gamma(L_4) = 6$.

## 5. CHAINS OF HIGHER ORDERS

We will denote by $G - S$ the graph obtained from a graph $G$ by removing a subset $S = \{v_1, v_2, \ldots, v_k\}$ of its nodes. When $k$ is small, we will simply write $G - v_1 - v_2 - \cdots - v_k$.

Let $P_r$ be the path with $r$ nodes. By *attaching* $P_r$ to a connected graph $G$, we will mean that an end-node of $P_r$ is identified with a node of $G$ to form a graph $H_r$ in which the subgraphs $P_r$ and $G$ are in the same component. We say $P_r$

is *added* to $G$ when the two end-nodes of $P_r$ are attached to different nodes of $G$. In this case, we assume that $G$ has more than two nodes. The resulting graph will be denoted by $J_r$. The nodes of $G$ used in the identification process will be called *nodes of attachment*.

The following lemmas will be useful in the material of this section.

**Lemma 7:** Let $u$ be the node of attachment of $H_r$. Then

$$\gamma(H_r) = \begin{cases} \gamma(G - u) & \text{if } r \text{ is even,} \\ \gamma(G) & \text{if } r \text{ is odd.} \end{cases}$$

**Proof:** Apply the reduction process to $H_r$ by deleting the edge of $P_r$ incident to $u$. The result follows immediately. ■

**Lemma 8:** Let $u$ and $v$ be the nodes of attachment of $J_r$. Then

$$\gamma(J_r) = \begin{cases} \gamma(G) + \gamma(G - u - v) & \text{if } r \text{ is even,} \\ \gamma(G - u) + \gamma(G - v) & \text{if } r \text{ is odd.} \end{cases}$$

**Proof:** The result follows easily by applying the reduction process by deleting an edge incident to one of the nodes of attachment and then using Lemma 7. ■

The edges of $P_{4,n}$ which join nodes of the upper and lower paths are called *link edges*, and the corresponding nodes are called *link nodes*. A *terminal edge* is a link edge which is incident to link nodes of valency 2. Also, we denote the $n^{\text{th}}$ Fibonacci number by $F_n$: $F_n = F_{n-1} + F_{n-2}$, $F_0 = F_1 = 1$.

**Theorem 5:** For $n, m, k \geqslant 1$,

(i) $\gamma(S_{4k+2,n}) = n + 1$,

(ii) $\gamma(S_{4k,n}) = F_{n+1}$,

(iii) $\gamma(S_{2k+1,2m+1}) = 0$

(iv) $\gamma(S_{2k+1,2m}) = m + 1$.

**Proof:**

(i) Apply Lemma 8 to $S_{4k+2,n}$. In this case $r$ is even. We get

$$\gamma(S_{4k+2,n}) = \gamma(S_{4k+2,n-1}) + \gamma(B_1), \tag{2}$$

where $B_1$ is the graph $S_{4k+2,n-2}$ with $P_{2k}$ attached to the ends of a terminal edge. Using Lemma 7, with $r$ even, we get $\gamma(B_1) = \gamma(B_2)$, where $B_2$ is $S_{4k+2,n-3}$ with $P_{2k}$ attached to the ends of a terminal edge. By repeated applications of the lemma, we get $\gamma(B_1) = 1$. Therefore, from Equation (2),

$$\gamma(S_{4k+2,n}) = \gamma(S_{4k+2,n-1}) + 1.$$

But $\gamma(S_{4k+2,1}) = \gamma(C_{4k+2}) = 2$. Therefore, we have

$$\gamma(S_{4k+2,n}) = n + 1.$$

(ii) Apply Lemma 8 to $S_{4k,n}$. Again $r$ is even, so we get

$$\gamma(S_{4k,n}) = \gamma(S_{4k,n-1}) + \gamma(A), \tag{3}$$

where $A$ is $S_{4k,n-2}$, with $P_{2k-1}$ attached to the ends of a terminal edge. Using Lemma 7, with $r$ odd, we get

$$\gamma(A) = \gamma(S_{4k,n-2}).$$

Hence, from Equation (3), we obtain

$$\gamma(S_{4k,n}) = \gamma(S_{4k,n-1}) + \gamma(S_{4k,n-2}).$$

Clearly $\gamma(S_{4k,1}) = 2 = F_2$ and $\gamma(S_{4k,2}) = 3 = F_3$. Therefore, we define

$$\gamma(S_{4k,0}) = 1 = F_1.$$

Hence, from Equation (3), we have

$$\gamma(S_{4k,n}) = F_{n+1}.$$

(iii) It can be easily verified that $S_{2k+1,2m+1}$ has an odd number of nodes $[2(2mk+k-m+1)+1]$. Hence, the result follows from Lemma 3.

(iv) First, we will label (in order) the link edges of $S_{2k+1,2m}$ with 1, 2, 3, ..., $2m+1$, beginning with a terminal edge. Let us apply the reduction process to $S_{2k+1,2m}$ by deleting an even labelled link edge. The graph $G''$ will contain two components; $A$, consisting of $S_{2k+1,i}$ with the chains $P_k$ and $P_{k-1}$ attached to the ends of a terminal edge, and $B$, consisting of $S_{2k+1,j}$ with $P_k$ and $P_{k-1}$ attached to the ends of a terminal edge. Clearly, $i + j = 2m - 2$ and both $i$ and $j$ will be even. It can be easily confirmed that $A$ will contain $2ik + 2k - i - 1$ nodes. Since this is odd, for all even values of $i$, we get

$$\gamma(A) = \gamma(B) = 0 \Rightarrow \gamma(G'') = 0.$$

Hence, no perfect matching contains an even (labelled) link edge. It follows that

$$\gamma(S_{2k+1,2m}) = \gamma(R_m),$$

where $R_m$ is the polygonal chain $P_{4k,m}$ obtained from $P_{4,m}$ by replacing each upper edge with $2k$ edges and each lower edge with $2k-2$ edges.

Apply Lemma 8 to $R_m$. This gives

$$\gamma(R_m) = \gamma(R_{m-1}) + \gamma(B), \tag{4}$$

where $B$ is the graph $R_{m-2}$ with $P_{2k}$ and $P_{2k-2}$ attached to the ends of a terminal edge. Hence, by an analysis similar to that used in establishing (i), we get

$$\gamma(R_m) = m + 1 = \gamma(S_{2k+1,2m}). \blacksquare$$

We now give bounds for general polygonal chains comprising $(2k+1)$-gons.

**Theorem 6:** For $m, k \geqslant 1$, $m + 1 \leqslant \gamma(P_{2k+1,2m}) \leqslant F_{m+1}$.

**Proof:** Let us construct $P_{2k+1,2m}$ from $P_{4,2m}$ by replacing the first pair of upper and lower edges with $P_{k+1}$ and $P_k$, respectively, the second pair by $P_k$ and $P_{k+1}$, respectively, the third pair by $P_{k+1}$ and $P_k$, respectively, and so on.

As we have shown above [(iii) of Theorem 5], no even link edge can belong to a perfect matching. Therefore, we can remove all the even labelled edges to obtain a graph which contains the same number of perfect matchings as $P_{2k+1,2m}$. In this case, the graph will be $S_{4k,m}$. Therefore,

$$\gamma(P_{2k+1,2m}) = \gamma(S_{4k,m}) = F_{m+1}, \text{ by (ii) of Theorem 5.} \tag{5}$$

When $P_{2k+1,2m}$ is the graph $S_{2k+1,2m}$, we get

$$\gamma(P_{2k+1,2m}) = \gamma(S_{2k+1,2m}) = m + 1, \text{ by (i) of Theorem 5.}$$

It can be seen from the proof of Theorem 5(iv) that, in the general case, the minimum value of $\gamma(B)$ in (4) is 1 and the maximum value is $\gamma(P_{2k+1,2m-2})$, and the result follows. ■

The following theorem is the long-chain analogue of Theorem 5.

**Theorem 7:** For $k \geqslant 1$, $m \geqslant 2$, and $n \geqslant 3$,

(i) $\gamma(L_{4k+2,n}) = 4$,

(ii) $\gamma(L_{4k,2m}) = \gamma(S_{4k,2m-1}) + \gamma(S_{4k,2m-3}) + 2 = F_{2m} + F_{2m-2} + 2$,

(iii) $\gamma(L_{4k,2m+1}) = \gamma(S_{4k,2m}) + \gamma(S_{4k,2m-1}) = F_{2m+1} + F_{2m-1}$,

(iv) $\gamma(L_{2k+1,2m-1}) = 0$,

(v) $\gamma(L_{2k+1,2m}) = 4$.

**Proof:**

(i) It can be easily confirmed that no perfect matching in $L_{4k+2,n}$ can contain a link edge. Therefore,

$$\gamma(L_{4k+2,n}) = \gamma(A) = 4,$$

where $A$ is the graph consisting of two disjoint cycles each with $2kn$ nodes.

(ii) and (iii), $k > 1$: Apply the reduction process of $L_{4k,r}$ by deleting the second upper edge (counting from the edge adjacent to a link edge) of a cell. Continue to apply the reduction process in the same nammer to both $G'$ and $G''$, but this time using the corresponding lower edge. The four resulting graphs will be the following: (1) $A_{r-1}$, consisting of the graphs $S_{4k,r-1}$ with $P_2$ attached to each end of a terminal edge and $P_{2k-2}$ attached to the ends of the other terminal edge; (2) $B_{r-1}$, consisting of the graph $S_{4k,r-1}$ with $P_2$ and $P_{k-2}$ attached to its two upper terminal nodes and $P_{2k-3}$ attached to the other end of the terminal edge adjacent to an edge of $P_{k-2}$ (note that $B_{r-1}$ will occur twice); and (3) $D_{r-1}$, the graph $S_{4k,r-1}$ with the odd chain $P_{2k-3}$ attached to the ends of a terminal edge.

It can be confirmed that:

1. $\gamma(A_{r-1}) = \gamma(S_{4k,r-3})$;

2. $\gamma(B_{r-1}) = \begin{cases} 1 & \text{if } r \text{ is even,} \\ 0 & \text{if } r \text{ is odd;} \end{cases}$

3. $\gamma(D_{r-1}) = \gamma(S_{4k,r-1})$.

For $k = 1$, the reduction process can be applied by deleting any upper edge. The graphs corresponding to $A_{r-1}$, $B_{r-1}$, and $D_{r-1}$ will be $S_{4k,r-1}$, $S_{4k,r-3}$ with $P_2$ attached to the two upper terminal nodes, and $S_{4k,r-3}$, respectively. Hence, for $k \geqslant 1$, we get

$$\gamma(L_{4k,r}) = \gamma(S_{4k,r-1}) + \gamma(S_{4k,r-3}) + \delta,$$

where $\delta = \begin{cases} 2 & \text{if } r \text{ is even,} \\ 0 & \text{if } r \text{ is odd.} \end{cases}$

The results (ii) and (iii) then follow from Theorem 5(ii).

(iv) It can easily be verified that $L_{2k+1,2m+1}$ has an odd number of nodes. Therefore, the result follows.

(v) This is similar to Theorem 5(iv). ∎

**Theorem 8:** For $m \geqslant 2$ and $k \geqslant 1$, $4 \leqslant \gamma(C_{2k+1,2m}) \leqslant 2(F_m + F_{m-2})$.

**Proof:** The proof is similar to that of Theorem 6. It follows by applying the reduction process to $C_{2k+1,2m}$, then using Equation (5) and Thoerem 7(v). ∎

Note that Theorems 3 and 4 are special cases of Theorems 6 and 8, respectively, when $k = 1$.

## REFERENCES


1. E. J. Farrell. "Introduction to Matching Polynomials." *J. Comb. Theory* **B** *27* (1979):75-86.
2. E. J. Farrell. "Matchings in Ladders." *ARS Combinatoria 6* (1978):153-161.
3. E. J. Farrell. "Matchings in Triangular Animals." *J. Comb. Info. and Systems Sciences 7*, no. 2 (1982):143-154.
4. E. J. Farrell & S. A. Wahid. "Matchings in Benzene Chains." *Discrete Applied Math. 7* (1984):45-54.
5. E. J. Farrell & S. A. Wahid. "Matchings in Pentagonal Chains." *Discrete Applied Math. 8* (1984):31-40.
6. F. Harary. *Graph Theory*. Reading, Mass.: Addison-Wesley, 1969.
7. F. Harary & E. Palmer. *Graphical Enumeration*. New York, London: Academic Press, 1973.


## ACKNOWLEDGMENT

The author thanks the referee for his invaluable suggestions, which helped to expand this paper to its present form.

◆◇◆◇◆