ON THE LEAST COMMON MULTIPLE OF SOME BINOMIAL COEFFICIENTS

HUGH M. EDGAR
San Jose State University, San Jose, CA 95192
(Submitted September 1984)

Let

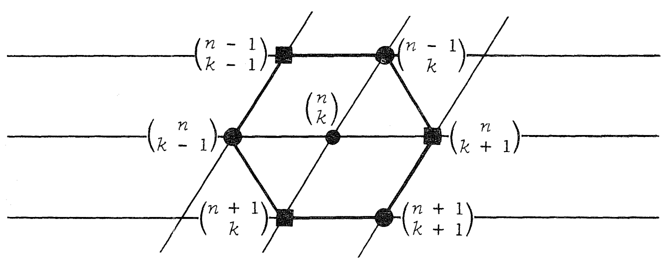
$$a = \binom{n-1}{k-1}\cdot\binom{n+1}{k},\quad b = \binom{n+1}{k}\cdot\binom{n}{k+1},\quad c = \binom{n}{k+1}\cdot\binom{n-1}{k-1},$$

$$d = \binom{n}{k-1}\cdot\binom{n+1}{k+1},\quad e = \binom{n+1}{k+1}\cdot\binom{n-1}{k},\ \text{and}\ f = \binom{n-1}{k}\cdot\binom{n}{k-1}.$$

We prove that

$$\text{L.C.M.}\{a, b, c\} = \text{L.C.M.}\{d, e, f\},$$

where L.C.M. denotes the least common multiple. The proof technique is due to the late Ernst Straus and rests upon elementary properties of the p-adic valuations of Q, the field of rational numbers. The geometry of the situation is indicated in the figure below.



Multiplying each of the quantities a through f by

$$\frac{k!(k+1)!(n-k)!(n-k+1)!}{(n-1)!n!}$$

produces the six corresponding quantities

$$(n+1)k(k+1),\ n(n+1)(n-k),\ k(n-k)(n-k+1),$$
$$n(n+1)k,\ (n+1)(n-k)(n-k+1),\ \text{and}\ k(k+1)(n-k).$$

Since $|\text{L.C.M.}\{\alpha, \beta\}|_p = \min\{|\alpha|_p, |\beta|_p\}$ for every p-adic valuation $|\ |_p$ of Q, the original problem is equivalent to proving that $m_1(n, k) = m_2(n, k)$ for all (finite) primes p, provided we define

$$m_1(n, k) = \min\{|(n+1)k(k+1)|_p,\ |n(n+1)(n-k)|_p,\ |k(n-k)(n-k+1)|_p\}$$

and

$$m_2(n, k) = \min\{|n(n+1)k|_p, \ |(n+1)(n-k)(n-k+1)|_p, \ |k(k+1)(n-k)|_p\}.$$

We first establish that $m_1(n, k) \geqslant m_2(n, k)$. In each of the three steps of this argument we make repeated use of the following standard facts concerning p-adic valuations of Q:

(1) the ultrametric inequality: $|\alpha + \beta|_p \leqslant \max\{|\alpha|_p, |\beta|_p\}$;

(2) $|\alpha + \beta|_p = \max\{|\alpha|_p, |\beta|_p\}$ if $|\alpha|_p \neq |\beta|_p$;

(3) $|z|_p \leqslant 1$, for every integer z and for every (finite) prime p;

(4) $|z|_p < 1$ if and only if the integer z is divisible by the prime p (equivalently, $|z|_p = 1$ if and only if the integer z is not divisible by the prime p).

We provide a detailed proof of the first step of the argument and then give somewhat abbreviated arguments for the remaining two steps.

Step 1. Assume that $|(n+1)k(k+1)|_p < m_2(n, k)$, that is,

(i) $|k+1|_p < |n|_p$,

(ii) $|k(k+1)|_p < |(n-k)(n-k+1)|_p$, and

(iii) $|n+1|_p < |n-k|_p$.

From (1) and (3), it follows that $|k+1|_p < 1$ so that, from (4), $p \mid k+1$. Since $(k, k+1) = 1$, it follows that $p \nmid k$, which can be rewritten using (4) as $|k|_p = 1$. From (iii) and (3), it follows that $|n+1|_p < 1 = |k|_p$ which, in conjunction with (2), allows us to conclude that

$$|n-k+1|_p = |(n+1)-k|_p = \max\{|n+1|_p, |k|_p\} = 1.$$

Going to (ii) and making use of the fact that $|k|_p = 1$ and $|n-k+1|_p = 1$, we get

$$|k(k+1)|_p = |k+1|_p < |(n-k)(n-k+1)|_p = |n-k|_p.$$

Finally

$$|n-k|_p = |(n+1)-(k+1)|_p \leqslant \max\{|n+1|_p, |k+1|_p\} < |n-k|_p,$$

from (1), and we have our desired contradiction.

Step 2. If $|n(n+1)(n-k)|_p < m_2(n, k)$, then we have

$$|n-k|_p < |k|_p, \ |n|_p < |n-k+1|_p, \text{ and } |n(n+1)|_p < |k(k+1)|_p.$$

Hence $|n-k+1| = |n+1| = 1$. Now,

$$|k|_p = |(n-k)-n|_p \leqslant \max\{|n-k|_p, |n|_p\} < |k|_p,$$

a contradiction. Here we made use of the fact that $|n|_p < |k(k+1)|_p \leqslant |k|_p$.

Step 3. If $|k(n-k)(n-k+1)|_p < m_2(n, k)$, then we have

$$|(n-k)(n-k+1)|_p < |n(n+1)|_p, \ |k|_p < |n+1|_p, \text{ and}$$

$$|n - k + 1|_p < |k + 1|_p.$$

Since $|n - k + 1| < 1$, we have $|n - k| = 1$, and so we get

$$|n - k + 1|_p < |n(n + 1)|_p \leqslant |n + 1|_p.$$

However,

$$|n - k + 1|_p = |(n + 1) - k|_p = \max\{|n + 1|_p, |k|_p\} = |n + 1|_p,$$

since $|k|_p < |n + 1|_p$. Hence, once again we have a contradiction.

Since $m_2(n, k) = m_1(-k - 1, -n - 1)$, and since $m_1(n, k) \geqslant m_2(n, k)$ has already been established, we can finish the proof using the following chain of inequalities:

$$\begin{aligned} m_1(n, k) \geqslant m_2(n, k) &= m_1(-k - 1, -n - 1) \geqslant m_2(-k - 1, -n - 1) \\ &= m_1(-(-n - 1) - 1, -(-k - 1) - 1) \\ &= m_1(n, k). \end{aligned}$$

Remarks: The result of this note can alternatively be deduced from the following previously established (see, respectively, [1], [2], and [3]) results:

(1) $\binom{n-1}{k} \cdot \binom{n}{k-1} \cdot \binom{n+1}{k+1} = \binom{n-1}{k-1} \cdot \binom{n}{k+1} \cdot \binom{n+1}{k}$

(2) $\text{G.C.D.}\left\{\binom{n-1}{k}, \binom{n}{k-1}, \binom{n+1}{k+1}\right\} = \text{G.C.D.}\left\{\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right\}$

where G.C.D. denotes the greatest common divisor.

(3) $xyz = \text{G.C.D.}\{x, y, z\} \cdot \text{L.C.M.}\{xy, yz, zx\}$, valid for arbitrary positive integers x, y, and z. A more involved result can be obtained using the fact (see [3]) that

$$xyz = \text{G.C.D.}\{x, y, z\} \cdot \text{L.C.M.}\{\text{G.C.D.}\{x, y\}, \text{G.C.D.}\{y, z\}, \text{G.C.D.}\{z, x\}\} \cdot \text{L.C.M.}\{x, y, z\}.$$

Finally, we ask whether such results have any combinatorial interpretation.

REFERENCES

1. V. E. Hoggatt, Jr., & Walter Hansell. "The Hidden Hexagon Squares." *The Fibonacci Quarterly* 9, no. 2 (1971):120, 133.
2. E. G. Straus. "On the Greatest Common Divisor of Some Binomial Coefficients." *The Fibonacci Quarterly* 11, no. 1 (1973):25-26.
3. Marlow Sholander. "Least Common Multiples and Highest Common Factors." *American Math Monthly* (1961):984.

◆◇◆◇◆