A CONGRUENCE RELATION FOR CERTAIN RECURSIVE SEQUENCES

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Recently, the first author [1] showed that

$$F_{n+5} \equiv F_n + F_{n-5} \pmod{10}, \tag{1}$$

where F_n is the n^{th} Fibonacci number, defined by $F_{n+1} = F_n + F_{n-1}$, $n \geqslant 2$, with $F_1 = F_2 = 1$. It was also shown [1] that this result generalizes to a sequence $\{S_n\}_1^\infty$ defined by

$$S_{n+1} = S_n + S_{n-1},\ n \geqslant 2,$$

with $S_1 = c$, $S_2 = d$, where c and d are nonnegative integers. The nonnegative restriction was imposed in order to guarantee that each member of the sequence is a positive number. However, the result is, in fact, valid for any integers c and d.

The purpose of this paper is to generalize (1) further. We will see that the role played by the integer 5 in (1) can, in the generalization, be played by any prime $p \geqslant 5$.

We begin by introducing a more general sequence $\{T_n\}_{-\infty}^{\infty}$ defined by

$$T_{n+1} = aT_n - bT_{n-1},\ \text{with } T_1 = c,\ T_2 = d, \tag{2}$$

where a, b, c, and d are integers with the restriction $b \neq 0$ (and exclusion of the trivial case where $c = d = 0$). We write $\{\alpha, \beta\}$ to denote the set of solutions of the quadratic equation $x^2 - ax + b = 0$. Two particular choices of c and d in (2) give rise to sequences $\{T_n\}$ of special interest to us. We denote these by $\{U_n\}_{-\infty}^{\infty}$ and $\{V_n\}_{-\infty}^{\infty}$, where

$$U_n = (\alpha^n - \beta^n)/(\alpha - \beta) \tag{3}$$

and

$$V_n = \alpha^n + \beta^n. \tag{4}$$

For $\{U_n\}$, $c = 1$ and $d = a$ while, for $\{V_n\}$, $c = a$ and $d = a^2 - 2b$. These sequences have been studied by Horadam [4]. [If $\alpha = \beta$, we replace (3) and (4) by the limiting forms $U_n = n\alpha^{n-1}$ and $V_n = 2\alpha^n$, respectively. Note that, in this case, $b = a^2/4$ and $\alpha = a/2$.] For the special case of (2) where $a = -b = 1$, the sequences $\{U_n\}$ and $\{V_n\}$ are, respectively, the Fibonacci and Lucas numbers for which (3) and (4) are the well-known Binet forms. We will write $\{L_n\}$ to denote the Lucas sequence.

Using $\alpha\beta = b$, we readily deduce from (3) and (4) that

$$U_{-n} = -b^{-n}U_n \tag{5}$$

and

$$V_{-n} = b^{-n}V_n.$$

We will require (5) later. We also need two lemmas connecting the sequences $\{U_n\}$ and $\{V_n\}$. The Fibonacci-Lucas forms of these (corresponding to $a = -b = 1$) are given in Hoggatt [3].

Lemma 1: For all integers k,

$$U_{k+1} - bU_{k-1} = V_k. \tag{6}$$

Proof: This is proved by induction or directly by using the generalized Binet forms (3) and (4).

Lemma 2: For all integers n and k,

$$U_{n+k} + b^k U_{n-k} = U_n V_k. \tag{7}$$

Proof: The proof may again be completed either by induction or by direct verification using (3) and (4). For the induction proof, we begin by verifying (7) for $n = 0$ and 1, with the aid of (5).

We generalize this last result to the sequence $\{T_n\}$ defined by (2).

Lemma 3: For all integers n and k,

$$T_{n+k} + b^k T_{n-k} = T_n V_k. \tag{8}$$

Proof: We show by induction that

$$T_n = dU_{n-1} - bcU_{n-2}, \tag{9}$$

and hence verify (8) directly from (7).

The results which we have obtained thus far are, in fact, valid when a, b, c, and d in (2) are real. However, for the divisibility results which follow, we require integer sequences; hence, we require a, b, c, and d to be integers. Also, in view of (5), we need to restrict $\{T_n\}$ to nonnegative n unless $|b| = 1$.

We now prove our first divisibility result.

Lemma 4: For any prime p,

$$V_p \equiv a \pmod{p}. \tag{10}$$

Proof: We need to treat the case $p = 2$ separately.

Since $V_2 = a^2 - 2b$,

$$V_2 - a = a(a-1) - 2b \equiv 0 \pmod{2}$$

for any choice of integers a and b.

If p is an odd prime,

$$a^p = (\alpha + \beta)^p = \sum_{r=0}^{p} \binom{p}{r} \alpha^{p-r}\beta^r.$$

From $\alpha\beta = b$, we obtain

$$\alpha^{p-r}\beta^r + \alpha^r\beta^{p-r} = b^r(\alpha^{p-2r} + \beta^{p-2r}).$$

and thus

$$a^p = V_p + \sum_{r=1}^{(p-1)/2} \binom{p}{r} b^r V_{p-2r}.$$

In the latter summation, we note that

$$\binom{p}{r} \equiv 0 \pmod{p}$$

for each r and the proof is completed by applying Fermat's theorem

$$a^p \equiv a \pmod{p}.$$

For the Fibonacci-Lucas case (where $a = -b = 1$), Lemma 4 yields

$$L_p \equiv 1 \pmod{p}$$

for any prime p. This special case, although not quoted explicitly, is easily deduced from congruence results for the Fibonacci numbers given in Hardy and Wright [2].

We now state the first of our main results.

Theorem 1: For all $n \geqslant p$ and all primes p,

$$T_{n+p} \equiv aT_n - bT_{n-p} \pmod{p}. \tag{11}$$

Proof: The proof follows from Lemmas 3 and 4 and Fermat's theorem. If $|b| = 1$, then (11) holds for all values of n.

Observe how the congruence relation (11) mimics the pattern of the recurrence relation (2).

To strengthen Theorem 1 for primes greater than 3, we first require:

Lemma 5: If $k \not\equiv 0 \pmod{3}$, then for all choices of a and b,

$$V_k \equiv a \pmod{2}. \tag{12}$$

Proof: In verifying (12) for all possible choices of a and b, it suffices to consider $\{a, b\} = \{0, 1\}$. If a is even and b is even or odd, V_k is even for all k and (12) holds. If a is odd and b is even, V_k is odd for all k and again (12) holds. Finally, if both a and b are odd, then V_k is even if and only if $k \equiv 0 \pmod{3}$, and the lemma is established.

Theorem 2: For all $n \geqslant p$, where p is any prime greater than 3,

$$T_{n+p} \equiv aT_n - bT_{n-p} \pmod{2p}. \tag{13}$$

[We note that (1) is the special case of (13) obtained by taking $p = 5$ and $a = -b = c = d = 1$.]

Proof: From the result of Theorem 1, it remains only to show that

$$T_{n+p} - aT_n + bT_{n-p} \equiv 0 \pmod{2}. \tag{14}$$

Using Lemma 3, the left side of (14) may be expressed as

$$(V_p - a)T_n + (b - b^p)T_{n-p}.$$

Observe that $b - b^p \equiv 0 \pmod 2$ and Lemma 5 shows that $V_p - a \equiv 0 \pmod 2$ for p any prime greater than 3, which completes the proof.

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