SIMSON'S FORMULA AND AN EQUATION OF DEGREE 24

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1. INTRODUCTION

In the January 23, 1985, issue of a local (Armidale) newspaper, L. Wilson, of Brisbane, announced that, if $x = F_n$, $y = F_{n+1}$ $(F_{n+2} = x + y)$ are successive numbers of the Fibonacci sequence $\{F_n\}$, then x, y $(>x)$ satisfy the equation of degree 24:

$$((x^5y - x^4y^2 - x^3y^3 + 3x^2y^4 - 3xy^5 + y^6)^2 - 4x^8 - 13x^4 - 1)^2 - 144x^{12} - 144x^8 - 36x^4 = 0. \tag{1}$$

This is a slight simplification of the equation announced three weeks earlier by him in the same newspaper.

Wilson offered no proof of his assertion.

It is the purpose of this paper to outline a proof of Wilson's result by analyzing the structure of (1).

We exclude $n = 0$ from our considerations to accord with the commencing Fibonacci number $F_1 = 1$ used by Wilson [although $x = 0$, $y = 1$ do satisfy (1)].

First, observe that Simson's formula for $\{F_n\}$, namely,

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1} \tag{2}$$

may be written

$$x^2 + xy - y^2 = 1, \; n \text{ odd}, \tag{3}$$

$$x^2 + xy - y^2 = -1, \; n \text{ even}. \tag{4}$$

Simson's formula will be the basic knowledge used in our proof.

2. PROOF OF THE ASSERTION

After a little elementary algebraic manipulation, the left-hand side of (1) factorizes as

$$(y^2A^2 - B_1^2)(y^2A^2 - B_2^2), \tag{5}$$

where

$$\begin{cases} A = x^5 - x^4y - x^3y^2 + 3x^2y^3 - 3xy^4 + y^5, \\ B_1 = 2x^4 - 3x^2 + 1, \\ B_2 = 2x^4 + 3x^2 + 1. \end{cases} \tag{6}$$

Numerical checking with small values of n establishes that the first factor in (5) vanishes for n odd, while the second factor in (5) vanishes for n even. This arithmetical evidence suggests that we may associate this first factor (and therefore B_1) with equation (3) and the second factor (and therefore B_2) with equation (4).

Accordingly, from (3), we have immediately $(x^2 - 1)^2 = (y^2 - xy)^2$ which, after tidying up and applying (3) again, gives us

$$B_1 = y(2y^3 - 4y^2x + 2x^2y - x + y). \tag{7}$$

Similarly,

$$B_2 = y(2y^3 - 4y^2x + 2x^2y + x - y). \tag{8}$$

Now $y - x$ is a factor of A, B_1, B_2. So (6) becomes

$$\begin{cases} A = (y - x)(y^4 - 2xy^3 + x^2y^2 - x^4) = (y - x)a, \\ B_1 = y(y - x)(2y^2 - 2xy + 1) = y(y - x)b_1, \\ B_2 = y(y - x)(2y^2 - 2xy - 1) = y(y - x)b_2. \end{cases} \tag{9}$$

Repeated multiplicative maneuvering with (3), followed by substitution in (9) and simplification, yields

$$b_1 = -a. \tag{10}$$

Appealing to B_2 and (4) by a similar argument, we find

$$b_2 = a. \tag{11}$$

From (9), it follows that (5) reduces to

$$y^4(y - x)^4(a^2 - b_1^2)(a^2 - b_2^2), \tag{12}$$

whence, by (10) and (11),

$$y^4(y - x)^4(a^2 - b_1^2)(a^2 - b_2^2) = 0, \tag{13}$$

i.e.,

$$(a^2 - b_1^2)(a^2 - b_2^2) = 0. \tag{14}$$

Thus, the validity of (1) is demonstrated.

Variations, perhaps simplifications, of the above reasoning no doubt exist.

3. REMARKS

Rearranging the four factors in (14) leads to

$$\{(a + b_1)(a - b_2)\}\{(a - b_1)(a + b_2)\} = 0. \tag{15}$$

By (10) and (11),

$$(a + b_1)(a - b_2) = 0. \tag{16}$$

Equation (16), which is of degree 8 in y, is thus also satisfied by successive pairs of Fibonacci numbers.

Even more ponderous and complicated equations of higher, but appropriate, degrees are suggested by (14). For instance,

$$(a^4 - b_1^4)(a^4 - b_2^4) = 0,$$

of degree 32 in y, is satisfied by the Fibonacci conditions.

Only the Fibonacci numbers provide the structure for (1). While similar patterns in (2), (3), and (4) exist for Lucas and Pell numbers, equations different from (1) would be germane to them.

Regarding the factors in (13) involving the fourth power, we remark that $y = 0$ if $n = -1$ (excluded), while $y - x = 0$ if $n = 1$, i.e., when $F_1 = F_2 = 1$.

Finally, we comment that (3) and (4) form the nucleus of a geometrical article on conics [2] by one of the authors, which was followed by an extension [1] by Bergum. One is prompted to speculate on the possibility of some arcane geometry of curves being obscured by the symbolism of (1).

REFERENCES

1. G. E. Bergum. "Addenda to Geometry of a Generalized Simson's Formula." *The Fibonacci Quarterly 22*, no. 1 (1984):22-28.
2. A. F. Horadam. "Geometry of a Generalized Simson's Formula." *The Fibonacci Quarterly* ***20***, no. 2 (1982):164-168.

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