THE EXISTENCE OF INFINITELY MANY k-SMITH NUMBERS

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1. INTRODUCTION

A Smith number has been defined by A.Wilansky [2] to be a composite number whose digit sum is equal to the sum of the digits of all its prime factors. Wilansky presents figures indicating that 360 Smith numbers occur among the first ten thousand positive integers, and asks whether there are infinitely many Smith numbers. Oltika and Wayland [1] have noted that relatively large Smith numbers are easily generated from primes whose digits are all 0's or 1's, but that only a small number of such primes are known.

We show in this paper that infinitely many Smith numbers do exist, using an approach that does not depend upon the primality of the integers used in the construction. This approach shows that, in fact, a much more general result holds.

Let *m* be a positive integer greater than 1. We denote the number of digits of *m* by N(m), the sum of the digits of *m* by S(m), and the sum of all the digits of all the prime factors of *m* by $S_p(m)$. It may be noted that $S_p(m) = S(m)$ if *m* is prime, and $S_p(m) = S_p(m_1) + S_p(m_2)$ if $m = m_1m_2 \quad (m_1, m_2 > 1)$.

Definition: Let *m* be a composite integer and *k* be any positive integer. *m* is a *k*-Smith number if $S_p(m) = kS(m)$.

An example of a 2-Smith number is $m = 104 = 2^3 \cdot 13$:

 $S_{v}(m) = 2 + 2 + 2 + 1 + 3 = 10 = 2(1 + 0 + 4) = 2S(m)$.

An example of a 3-Smith number is $402 = 2 \cdot 3 \cdot 67$. Among the positive integers less than 1000, there are $47 \ k$ -Smith numbers for k = 1 (see [2] for additional information on the distribution of Smith numbers), twenty-one for k = 2, three for k = 3, and one k-Smith number for each of k = 7, 9, and 14.

The principal result of this paper is that infinitely many k-Smith numbers exist for every positive integer k.

2. SOME FUNDAMENTAL PROPERTIES

First, we obtain an upper bound on $S_p(m)$ which does not involve the specific prime factors of m.

Theorem 1: If p_1, \ldots, p_r are prime numbers, not necessarily distinct, and if $m = p_1 p_2 \cdots p_r$, then $S_p(m) < 9N(m) - .54r$.

Proof: Let $b_i = N(p_i) - 1$, i = 1, 2, ..., r, and $b = b_1 + \cdots + b_r$. Now, the sum of the digits of a prime is not a multiple of 9, so

$$S(p_i) \leq 9N(p_i) - 1 = 9b_i + 8.$$

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We partition the prime factors of *m* into 9 disjoint classes by means of the following: Let c_i be defined by $S(p_i) = 9b_i + c_i$, $c_i \leq 8$, $i = 1, 2, \ldots, r$, n_0 be the number of integers i $(1 \leq i \leq r)$ for which c_i is negative, and n_j be the number of integers i $(1 \leq i \leq r)$ such that $c_i = j$, for $1 \leq j \leq 8$. Then,

$$S_p(m) = \sum_{i=1}^{r} S(p_i) = \sum_{i=1}^{r} (9b_i + c_i) = 9b + \sum_{j=1}^{8} jn_j + \sum c_i,$$

where this last sum is over the n_0 values of i for which $c_i < 0$ (note that $c_i \neq 0$ for any i). Since the last sum is less than or equal to $-n_0$, we have

$$S_p(m) \leq 9b + \sum_{j=1}^{8} jn_j - n_0.$$
 (1)

Now, $S(p_i) = 9b_i + c_i$ implies, for $c_i < 0$, that $p_i > 10^{b_i}$, and, for $1 \le c_i \le 8$, that

$$p_i \ge (c_i + 1) \cdot 10^{b_i} - 1 \ge (c_i + 9/10) \cdot 10^{b_i}$$
, if $b_i \ge 0$

(i.e., unless p_i is one of the primes 2, 3, 5, or 7), and

$$p_i = c_i 10^{b_i}$$
, if $b_i = 0$.

It follows that

so

$$m = p_1 p_2 \cdots p_r$$

$$\geq (1.9)^{n_1} (2)^{n_2} (3)^{n_3} (4.9)^{n_4} (5)^{n_5} (6.9)^{n_6} (7)^{n_7} (8.9)^{n_6} \cdot 10^b.$$

Rewriting *m* as $a \cdot 10^{N(m)-\circ}$, for some rational number $1 \le a \le 10$, and taking logarithms, base 10, we have

 $\log a + N(m) - 1 \ge n_1 \log 1.9 + \dots + n_8 \log 8.9 + b,$ $9N(m) \ge 9b + n_1(9 \log 1.9) + \dots + n_8(9 \log 8.9) + 9(1 - \log a).$

For each integer j (1 \leq j \leq 8), we find that the coefficient of n_j is greater than j + .54. Hence,

$$9b + \sum_{j=1}^{8} n_{j}(j + .54) < 9N(m),$$

that is,
$$9b + \sum_{j=1}^{8} jn_{j} < 9N(m) - .54(n_{1} + \dots + n_{8}).$$
 (2)

Combining (1) and (2), we have

$$S_p(m) < 9N(m) - .54(n_0 + n_1 + \cdots + n_8) - .46n_0 \le 9N(m) - .54r. Q.E.D.$$

We now state without proof a fact that is surely well known but which we have not found in the literature. The proof follows readily upon writing $t\ {\rm as}$

$$\sum_{i=0}^{k} a_i 10^{i} \quad (k < n).$$

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Theorem 2: If $10^n - 1$ is multiplied by a positive integer $t \le 10^n - 1$, the digit sum of the product is $9n \ (n \ge 1)$.

3. *k*-SMITH NUMBERS

Theorem 3: Let c be any nonnegative integer. There exist infinitely many integers M for which $S_p(M) = S(M) + c$.

Proof: Let $m = 10^n - 1$, $n \ge 2$. Since $3^2 | m$, m has at least three prime factors, so, by Theorem 1, $S_p(m) \le 9N(m) - 2 = 9n - 2$. Let $h = 9n - S_p(m) \ge 2$. We define

 $T = \{2, 3, 4, 5, 8, 7, 15\},$

making

 $\{S_p(t) \mid t \in T\} = \{2, 3, 4, 5, 6, 7, 8\}$

a complete residue system (mod 7).

Since c nonnegative implies that $h + c \ge 2$, there exists an integer $t \in T$ such that $S_p(t) = h + c - 7b$ for some nonnegative integer b. We now consider the product $M = t(10^n - 1) \cdot 10^b$.

Noting that a power of 10 times a number has the same digit sum as the number, we have, by Theorem 2, S(M) = 9 . Hence,

$$S_p(M) = S_p(t) + S_p(10^n - 1) + S_p(10^p)$$

= (h + c - 7b) + (9n - h) + 7b
= 9n + c
= S(M) + c.

This secures the theorem, since each n determines a unique M.

Corollary: There exist infinitely many k-Smith numbers for each positive integer k.

Proof: Let k and n be positive integers, and M be defined as in Theorem 3. We need only choose c equal to $(k - 1) \cdot 9n = (k - 1)S(M)$; thus,

$$S_p(M) = S(M) + (k - 1)S(M) = kS(M).$$

When k = 1, we have, of course, a Smith number for each integer $n \ge 2$ [actually, for $n \ge 1$, since $S(t(10^1 - 1)) = 9$ for each $t \in T$].

The following algorithm for constructing k-Smith numbers is implicit in the proofs of Theorem 3 and the Corollary.

Algorithm:

- 1. Let $n \ge 2$ and factor $m = 10^n 1$;
- 2. Compute $S_p(m)$ and set $h = 9n S_p(m)$;
- 3. Solve x = h + (k 1)9n 7b, $2 \le x \le 8$, and find $t \in T$ such that $S_p(t) = x$.
- 4. $M = t(10^{n} 1) \cdot 10^{b}$ is a k-Smith number.

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Example 1: A Smith number (k = 1).

Let $m = 10^6 - 1 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$ (n = 6 has been chosen arbitrarily). $S_p(m) = 32; h = 54 - 32 = 22.$ x = 22 - 7b implies that x = 8 and b = 2. S(t) = 8 implies t = 15. Hence, $M = 15(10^6 - 1) \cdot 10^2 = 1,499,998,500$ is a Smith number. $[S_p(M) = S_p(3) + S_p(5) + 32 + 14 = 54 = S(M).]$

Example 2: A 6-Smith number.

Let $m = 10^2 - 1 = 3^2 \cdot 11$. $S_p(m) = 8$; h = 18 - 8 = 10. x = 100 - 7b implies that x = 2 and b = 14. S(t) = 2 implies t = 2. Hence, $M = 2(10^2 - 1) \cdot 10^{14} = 2^{15} \cdot 3^2 \cdot 11 \cdot 5^{14}$ is a 6-Smith number. $[S_p(M) = 30 + 6 + 2 + 70 = 108 = 6S(M).]$

4. SOME REMAINING QUESTIONS

Thus far, it has become clear that there exist infinitely many integers m for which $S_p(m)$ far exceeds S(m). Now, it is conceivable that the "opposite" relationship may also hold. If, in fact, one examines the composite integers m < 1000, one finds that $S_p(m) < S(m)$ for approximately 37% of these values. We make the following definition.

Definition: Let *m* be a composite integer and *k* be any positive integer. *m* is a k^{-1} -Smith number if $S_p(m) = k^{-1}S(m)$. [That is, if $kS_p(m) = S(m)$.]

There are nine k^{-1} -Smith numbers ($k \ge 1$) less than 1000—all 2⁻¹-Smith numbers. The smallest is 88:

$$S_p(88) = S_p(2^3 \cdot 11) = 8 - \frac{1}{2}S(88),$$

and an example of a 3^{-1} -Smith number is 19,998. The largest k for which we have found a k^{-1} -Smith number is 6:

 $3^2 \cdot 11 \cdot 101 \cdot (100003)^2 = 99,995,999,489,991$

is a 6⁻¹-Smith number.

The following argument shows that it is possible that k^{-1} -Smith numbers exist for larger integers k.

Suppose that, for some integer $n \ge 2$, $10^n + 1$ is a prime (this implies that n is a power of 2 and $n \ge 1024$; see [3, p. 63]).

Let $\binom{t}{a}$ be the largest binomial coefficient in the expansion of $(10^n + 1)^t$, t any integer such that $\binom{t}{a} < 10^4 - 1$, and let $m = 9999(10^n + 1)^t$. The restriction on t assures that the coefficient of 10^{jn} ($0 \le j \le t$) in the expansion of m has digit sum 36, by Theorem 2. Since $9999\binom{t}{a}10^{jn} < 10^{(j+1)n}$, S(m) = 36(t + 1), and it is clear that $S_p(m) = 10 + 2t$. Thus, for t = 1, 3, 7, and 13, m is a k^{-1} -Smith number for k = 6, 9, 12, and 14, respectively. However, at present no primes of the form $10^n + 1$, other than 11 and 101, are known.

Accordingly, we pose the following questions: Is there a k^{-1} -Smith number for every integer k? If not, what is the largest k for which k^{-1} -Smith numbers

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exist? Do there exist infinitely many 2^{-1} -Smith numbers? Do there exist infinitely many k^{-1} -Smith numbers for any k > 2?

We conjecture that the answer to each of the last two questions is "yes."

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