# THE EXISTENCE OF INFINITELY MANY $k$-SMITH NUMBERS 

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A Smith number has been defined by A. Wilansky [2] to be a composite number whose digit sum is equal to the sum of the digits of all its prime factors. Wilansky presents figures indicating that 360 Smith numbers occur among the first ten thousand positive integers, and asks whether there are infinitely many Smith numbers. Oltika and Wayland [1] have noted that relatively large Smith numbers are easily generated from primes whose digits are all 0's or $1^{\prime} \mathrm{s}$, but that only a small number of such primes are known.

We show in this paper that infinitely many Smith numbers do exist, using an approach that does not depend upon the primality of the integers used in the construction. This approach shows that, in fact, a much more general result holds.

Let $m$ be a positive integer greater than 1 . We denote the number of digits of $m$ by $N(m)$, the sum of the digits of $m$ by $S(m)$, and the sum of all the digits of all the prime factors of $m$ by $S_{p}(m)$. It may be noted that $S_{p}(m)=S(m)$ if $m$ is prime, and $S_{p}(m)=S_{p}\left(m_{1}\right)+S_{p}\left(m_{2}\right)$ if $m=m_{1} m_{2}\left(m_{1}, m_{2}>1\right)$.

Definition: Let $m$ be a composite integer and $k$ be any positive integer. $m$ is a $k$-Smith number if $S_{p}(m)=k S(m)$.

An example of a 2-Smith number is $m=104=2^{3} \cdot 13$ :

$$
S_{p}(m)=2+2+2+1+3=10=2(1+0+4)=2 S(m)
$$

An example of a 3 -Smith number is $402=2 \cdot 3 \cdot 67$. Among the positive integers less than 1000, there are $47 k$-Smith numbers for $k=1$ (see [2] for additional information on the distribution of Smith numbers), twenty-one for $k=2$, three for $k=3$, and one $k$-Smith number for each of $k=7,9$, and 14 .

The principal result of this paper is that infinitely many $k$-Smith numbers exist for every positive integer $k$.

## 2. SOME FUNDAMENTAL PROPERTIES

First, we obtain an upper bound on $S_{p}(m)$ which does not involve the specific prime factors of $m$.

Theorem 1: If $p_{1}, \ldots, p_{r}$ are prime numbers, not necessarily distinct, and if $m=p_{1} p_{2} \cdots p_{r}$, then $S_{p}(m)<9 N(m)-.54 r$.

Proof: Let $b_{i}=N\left(p_{i}\right)-1, i=1,2, \ldots, r$, and $b=b_{1}+\cdots+b_{r}$. Now, the sum of the digits of a prime is not a multiple of 9 , so

$$
S\left(p_{i}\right) \leqslant 9 N\left(p_{i}\right)-1=9 b_{i}+8
$$

## the existence of infinitely many $k$-SMITH Numbers

We partition the prime factors of $m$ into 9 disjoint classes by means of the following: Let $c_{i}$ be defined by $S\left(p_{i}\right)=9 b_{i}+c_{i}, c_{i} \leqslant 8, i=1,2, \ldots, r, n_{0}$ be the number of integers $i(1 \leqslant i \leqslant r)$ for which $c_{i}$ is negative, and $n_{j}$ be the number of integers $i(1 \leqslant i \leqslant r)$ such that $c_{i}=j$, for $1 \leqslant j \leqslant 8$. Then,

$$
S_{p}(m)=\sum_{i=1}^{r} S\left(p_{i}\right)=\sum_{i=1}^{r}\left(9 b_{i}+c_{i}\right)=9 b+\sum_{j=1}^{8} j n_{j}+\sum c_{i}
$$

where this last sum is over the $n_{0}$ values of $i$ for which $c_{i}<0$ (note that $c_{i} \neq$ 0 for any $i$ ). Since the last sum is less than or equal to $-n_{0}$, we have

$$
\begin{equation*}
S_{p}(m) \leqslant 9 b+\sum_{j=1}^{8} j n_{j}-n_{0} \tag{1}
\end{equation*}
$$

Now, $S\left(p_{i}\right)=9 b_{i}+c_{i}$ implies, for $c_{i}<0$, that $p_{i}>10^{b_{i}}$, and, for $1 \leqslant c_{i} \leqslant 8$, that

$$
p_{i} \geqslant\left(c_{i}+1\right) \cdot 10^{b_{i}}-1 \geqslant\left(c_{i}+9 / 10\right) \cdot 10^{b_{i}} \text {, if } b_{i}>0
$$

(i.e., unless $p_{i}$ is one of the primes $2,3,5$, or 7 ), and

$$
p_{i}=c_{i} 10^{b_{i}}, \text { if } b_{i}=0
$$

It follows that

$$
\begin{aligned}
m & =p_{1} p_{2} \cdots p_{r} \\
& \geqslant(1.9)^{n_{1}}(2)^{n_{2}}(3)^{n_{3}}(4.9)^{n_{4}}(5)^{n_{5}}(6.9)^{n_{6}}(7)^{n_{7}}(8.9)^{n_{8}} \cdot 10^{b} .
\end{aligned}
$$

Rewriting $m$ as $a \cdot 10^{N(m)-0}$, for some rational number $1 \leqslant \alpha<10$, and taking logarithms, base 10, we have

$$
\log a+N(m)-1 \geqslant n_{1} \log 1.9+\cdots+n_{8} \log 8.9+b,
$$

so

$$
9 N(m) \geqslant 9 b+n_{1}(9 \log 1.9)+\cdots+n_{8}(9 \log 8.9)+9(1-\log a) .
$$

For each integer $j(1 \leqslant j \leqslant 8)$, we find that the coefficient of $n_{j}$ is greater than $j+.54$. Hence,

$$
9 b+\sum_{j=1}^{8} n_{j}(j+.54)<9 N(m),
$$

that is,

$$
\begin{equation*}
9 b+\sum_{j=1}^{8} j n_{j}<9 N(m)-.54\left(n_{1}+\cdots+n_{8}\right) \tag{2}
\end{equation*}
$$

Combining (1) and (2), we have

$$
S_{p}(m)<9 N(m)-.54\left(n_{0}+n_{1}+\cdots+n_{8}\right)-.46 n_{0} \leqslant 9 N(m)-.54 r \cdot \text { Q.E.D. }
$$

We now state without proof a fact that is surely well known but which we have not found in the literature. The proof follows readily upon writing $t$ as

$$
\sum_{i=0}^{k} a_{i} 10^{i} \quad(k<n)
$$

Theorem 2: If $10^{n}-1$ is multiplied by a positive integer $t \leqslant 10^{n}-1$, the digit sum of the product is $9 n(n \geqslant 1)$.

## 3. $k$-SMITH NUMBERS

Theorem 3: Let $c$ be any nonnegative integer. There exist infinitely many integers $M$ for which $S_{p}(M)=S(M)+c$.

Proof: Let $m=10^{n}-1, n \geqslant 2$. Since $3^{2} \mid m, m$ has at least three prime factors, so, by Theorem $1, S_{p}(m) \leqslant 9 N(m)-2=9 n-2$. Let $h=9 n-S_{p}(m) \geqslant 2$. We define

$$
T=\{2,3,4,5,8,7,15\},
$$

making

$$
\left\{S_{p}(t) \mid t \in T\right\}=\{2,3,4,5,6,7,8\}
$$

a complete residue system (mod 7).
Since $c$ nonnegative implies that $h+c \geqslant 2$, there exists an integer $t \in T$ such that $S_{p}(t)=h+c-7 b$ for some nonnegative integer $b$. We now consider the product $M=t\left(10^{n}-1\right) \cdot 10^{b}$.

Noting that a power of 10 times a number has the same digit sum as the number, we have, by Theorem $2, S(M)=9$. Hence,

$$
\begin{aligned}
S_{p}(M) & =S_{p}(t)+S_{p}\left(10^{n}-1\right)+S_{p}\left(10^{b}\right) \\
& =(h+c-7 b)+(9 n-h)+7 b \\
& =9 n+c \\
& =S(M)+c .
\end{aligned}
$$

This secures the theorem, since each $n$ determines a unique $M$.
Corollary: There exist infinitely many $k$-Smith numbers for each positive integer $k$.

Proof: Let $k$ and $n$ be positive integers, and $M$ be defined as in Theorem 3 . We need only choose $c$ equal to $(k-1) \cdot 9 n=(k-1) S(M)$; thus,

$$
S_{p}(M)=S(M)+(k-1) S(M)=k S(M) .
$$

When $k=1$, we have, of course, a Smith number for each integer $n \geqslant 2$ [actually, for $n \geqslant 1$, since $S\left(t\left(10^{1}-1\right)\right)=9$ for each $\left.t \in T\right]$.

The following algorithm for constructing $k$-Smith numbers is implicit in the proofs of Theorem 3 and the Corollary.

## Algorithm:

1. Let $n \geqslant 2$ and factor $m=10^{n}-1$;
2. Compute $S_{p}(m)$ and set $h=9 n-S_{p}(m)$;
3. Solve $x=h+(k-1) 9 n-7 b, 2 \leqslant x \leqslant 8$, and find $t \in T$ such that $S_{p}(t)=x$.
4. $M=t\left(10^{n}-1\right) \cdot 10^{b}$ is a $k$-Smith number.

Example 1: A Smith number $(k=1)$.
Let $m=10^{6}-1=3^{3} \cdot 7 \cdot 11 \cdot 13 \cdot 37 \quad(n=6$ has been chosen arbitrarily). $S_{p}(m)=32 ; h=54-32=22 . x=22-7 b$ implies that $x=8$ and $b=2 . \quad S(t)$ $=8$ implies $t=15$. Hence, $M=15\left(10^{6}-1\right) \cdot 10^{2}=1,499,998,500$ is a Smith number. $\left[S_{p}(M)=S_{p}(3)+S_{p}(5)+32+14=54=S(M).\right]$

Example 2: A 6-Smith number.
Let $m=10^{2}-1=3^{2} \cdot 11 . \quad S_{p}(m)=8 ; h=18-8=10 . \quad x=100-7 b$ implies that $x=2$ and $b=14$. $S(t)=2$ implies $t=2$. Hence, $M=2\left(10^{2}-1\right) \cdot 10^{14}=$ $2^{15} \cdot 3^{2} \cdot 11 \cdot 5^{14}$ is a 6 -Smith number. $\left[S_{p}(M)=30+6+2+70=108=6 S(M).\right]$

## 4. SOME REMAINING QUESTIONS

Thus far, it has become clear that there exist infinitely many integers $m$ for which $S_{p}(m)$ far exceeds $S(m)$. Now, it is conceivable that the "opposite" relationship may also hold. If, in fact, one examines the composite integers $m<1000$, one finds that $S_{p}(m)<S(m)$ for approximately $37 \%$ of these values. We make the following definition.

Definition: Let $m$ be a composite integer and $k$ be any positive integer. $m$ is a $k^{-1}$-Smith number if $S_{p}(m)=k^{-1} S(m)$. [That is, if $k S_{p}(m)=S(m)$.]

There are nine $k^{-1}$-Smith numbers $(k>1)$ less than 1000 -all $2^{-1}$-Smith numbers. The smallest is 88:

$$
S_{p}(88)=S_{p}\left(2^{3} \cdot 11\right)=8-\frac{1}{2} S(88),
$$

and an example of a $3^{-1}$-Smith number is 19,998. The largest $k$ for which we have found a $k^{-1}-$ Smith number is 6 :

$$
3^{2} \cdot 11 \cdot 101 \cdot(100003)^{2}=99,995,999,489,991
$$

is a $6^{-1}$-Smith number.
The following argument shows that it is possible that $k^{-1}$-Smith numbers exist for larger integers $k$.

Suppose that, for some integer $n>2,10^{n}+1$ is a prime (this implies that $n$ is a power of 2 and $n \geqslant 1024$; see [3, p. 63]).

Let $\binom{t}{a}$ be the largest binomial coefficient in the expansion of $\left(10^{n}+1\right)^{t}$, $t$ any integer such that $\binom{t}{a}<10^{4}-1$, and let $m=9999\left(10^{n}+1\right)^{t}$. The restriction on $t$ assures that the coefficient of $10^{j n}(0 \leqslant j \leqslant t)$ in the expansion of $m$ has digit sum 36, by Theorem 2. Since $9999\binom{t}{a} 10^{j n}<10^{(j+1) n}, S(m)=36(t+$ 1 ), and it is clear that $S_{p}(m)=10+2 t$. Thus, for $t=1,3,7$, and $13, m$ is a $k^{-1}$-Smith number for $k=6,9,12$, and 14 , respectively. However, at present no primes of the form $10^{n}+1$, other than 11 and 101 , are known.

Accordingly, we pose the following questions: Is there a $k^{-1}$-Smith number for every integer $k$ ? If not, what is the largest $k$ for which $k^{-1}$-Smith numbers
exist? Do there exist infinitely many $2^{-1}$-Smith numbers? Do there exist infinitely many $k^{-1}$-Smith numbers for any $k>2$ ?

We conjecture that the answer to each of the last two questions is "yes."

## REFERENCES

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