REPRESENTING $\binom{2n}{n}$ AS A SUM OF SQUARES

NEVILLE ROBBINS San Francisco State University, San Francisco, CA 94132 (Submitted March 1985)

INTRODUCTION

A well-known theorem of Lagrange [4, p. 302] states that every natural number can be represented as a sum of at most four squares. For each integer, k, such that $1 \le k \le 4$, let S_k be the set of natural numbers, n, such that $\binom{2n}{n}$ is a sum of k (but not fewer) squares. We show that S_1 is empty, $S_2 = \{1, 3\}$, while S_3 and S_4 are both infinite.

PRELIMINARIES

Let p denote a prime.

Definition 1: $o_p(n) = k \text{ if } p^k | n, p^{k+1} \nmid n$

Definition 2:
$$t_p(n) = \sum_{i=0}^r a_i$$
 if $n = \sum_{i=0}^r a_i p^i$, with $0 \le a_i < p$ for each i .

$$o_p(ab) = o_p(a) + o_p(b) \tag{1}$$

$$o_p(n!) = \frac{n - t_p(n)}{p - 1}$$
 (2)

$$o_p(\binom{n}{k}) = \frac{t_p(k) + k_p(n-k) - t_p(n)}{p-1}$$
(3)

$$t_p(ap^j) = t_p(a) \text{ for all } a, j$$
(4)

$$\mathcal{O}_2\left(\binom{2n}{n}\right) = t_2(n) \tag{5}$$

$$n \neq a^2 + b^2 + c^2$$
 iff $n = 2^{2k} (8m + 7)$ with $k \ge 0, m \ge 0$ (6)

$$n \neq a^2 + b^2$$
 iff there is a prime, p, such that

$$p \equiv 3 \pmod{4}$$
 and $o_p(n)$ is odd. (7)

Remarks: (1) follows from Definition 1. (2) is [2, p. 131, Problem 7]. (3) follows from (1) and (2). (4) follows from Definition 2. (5) follows from (3) and (4). (6) is stated in [4, p. 311]. (7) is [4, p. 299, Theorem 366]. $t_2(n)$ is denoted $\#_1(n)$ in [5].

1987]

REPRESENTING $\binom{2n}{n}$ AS A SUM OF SQUARES

THE MAIN THEOREMS

Theorem 1: If $n \neq 1$, 3, then there is a prime, p, such that $p \equiv 3 \pmod{4}$ and n .

Proof: Breusch [1] proved the conclusion for $n \ge 7$. If n = 2, then p = 3; if $4 \le n \le 6$, then p = 7.

Theorem 2: S_1 is empty; $S_2 = \{1, 3\}$.

Proof: If $2 \le n , then <math>2n < 2p$, so $o_p\left(\binom{2n}{n}\right) = 1$. Therefore, (7) and Theorem 1 imply $S_1 \cup S_2 \subseteq \{1, 3\}$. Since

$$\binom{2}{1}$$
 = 1² + 1², and $\binom{6}{3}$ = 4² + 2²

the conclusion now follows.

Remark: That S_1 is empty also follows from the theorem of P. Erdos [3], which states that $\binom{n}{k}$ is not a power if k > 3.

Definition 3: If $n = 2^k m$, $k \ge 0$, m odd, then f(n) is the least positive residue of $m \pmod{8}$.

Lemma 1: If *m* is odd, then $f(m) \equiv m \pmod{8}$.

Proof: The proof follows from the hypothesis and Definition 3.

Lemma 2: If $f(a) \equiv f(b) \pmod{8}$, then f(a) = f(b).

Proof: The proof follows from the hypothesis and Definition 3.

Lemma 3: $f(ab) \equiv f(a)f(b) \pmod{8}$.

Proof: Let $a = 2^{c}j$, $b = 2^{d}k$, with $c \ge 0$, $d \ge 0$, jk odd. Lemma 1 implies

 $f(jk) \equiv jk \equiv f(j)f(k) \pmod{8}$.

Now $f(ab) = f(2^{c+d}jk) = f(jk)$, while f(a)f(b) = f(j)f(k), so

 $f(ab) \equiv f(a)f(b) \pmod{8}$.

Lemma 4: If f(b) = 1, then f(ab) = f(a).

Proof: The proof follows from the hypothesis and Lemmas 3 and 2.

Lemma 5: $f(n^2) = 1$.

Proof: If $n = 2^k m$, $k \ge 0$, m odd, then $f(n^2) = f(2^{2k}m^2) = f(m^2)$. Now, Lemma 1 implies $f(m^2) \equiv m^2 \equiv 1 \pmod{8}$. But f(1) = 1, so we have $f(n^2) \equiv f(1) \pmod{8}$. Now, Lemma 2 implies $f(n^2) = f(1) = 1$.

Lemma 6: $f\left(\binom{2n}{n}\right) = f((2n)!)$

Proof: The proof follows from Lemmas 4 and 5, since $(2n)! = \binom{2n}{n} (n!)^2$.

[Feb.

30

REPRESENTING $\binom{2n}{n}$ As a sum of squares

Definition 4: Let g(n) = f(n!).

Table 1 lists g(n) and $t_2(n)$ for each n such that $1 \le n \le 200$.

Table 1

$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	п	g(n)	$t_2(n)$	n	g(n)	$t_2(n)$	n	g(n)	$t_2(n)$	n	g(n)	$t_2(n)$
2115213102341525333253541035515354431547410413154145725515105141553565256731065415654733577410775157158315834108541587593259151091516052115361351111616153127262551127316253133363361137416473153465321155165341631114741647331534653211551677520527073120741707421137114	1	1	1	51	5	4	101	1	4	151	7	5
3325354103551535443154741041315414572551510514155356525673106541565473357741077515715831583410854158759325915109151591610726074110751605311536135111161615312726255112731625313336431114741637414536431114716634153465321155167751631695311917168339113711412175171552233	2	1	1	52	1	3	102	3	4	152	5	3
431547410413154145725515105141553565256731065415535733577410775157159325915109151591610726074110751605211536135111161615312726255112731625313363361137416473145364311147416473153465321161416614163166321161416473163166321187516775813268121187416414173711412175171552233 <t< td=""><td>3</td><td>-3</td><td>2</td><td>53</td><td>5</td><td>4</td><td>103</td><td>5</td><td>5</td><td>153</td><td>5</td><td>4</td></t<>	3	-3	2	53	5	4	103	5	5	153	5	4
572551510514155356525673106541565473357741077515715831583410854158759325915109151591610726074110751605211536135111161615312726255112731625313336336113741647314536431114741661415346532116141661416316632116141661417369531191716934205270731207417074211375741217517455223 <t< td=""><td>4</td><td>3</td><td>1</td><td>54</td><td>7</td><td>4</td><td>104</td><td>1</td><td>3</td><td>154</td><td>1</td><td>4</td></t<>	4	3	1	54	7	4	104	1	3	154	1	4
6525673106541565473357741077515715831583410854158759325915109151591610726074110751605211536135111161615312726255112731625313363361137416374145364311147416614153465321161416614163166321161416614173267131175516775813268121187516833911369531191716934205270731207417074211<	5	7	2	55	1	5	105	1	4	155	3	5
73357741077515715831583410854158759325915109151591610726074110751605211536135111161615312726255112731625313363361137416374145364311147416473153465321155516534163166321161416614173267131175516775813268121187516833911369531191716934205270731207417074211371141217517155223	6	5	2	56	7	3	106	5	4	156	5	4
831583410854158759325915109151591610726074110751605211536135111161615312726255112731625313336336113741637414536431114741661415346532116141661416316632116141661417326713117551677581326812118751683391136953119171693420527073120741707421137114121751715522337574125361753624 <td>7</td> <td>3</td> <td>3</td> <td>57</td> <td>7</td> <td>4</td> <td>107</td> <td>7</td> <td>5</td> <td>157</td> <td>1</td> <td>5</td>	7	3	3	57	7	4	107	7	5	157	1	5
9325915109151591610726074110751605211536135111161615312726255112731625313336336113741637414536431114741647315346532115551653416316632116141661417326713117551677581326812118751683391136953119171693420527073120741707421137114121751715522337212122351727421137574125361753624 <td>8</td> <td>3</td> <td>1</td> <td>58</td> <td>3</td> <td>4</td> <td>108</td> <td>5</td> <td>4</td> <td>158</td> <td>/</td> <td>5</td>	8	3	1	58	3	4	108	5	4	158	/	5
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	9	3	2	59	1	5	109	1	5	159	1	6
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	10	/	2	60	/	4		/	5	160	5	2
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	12	5	3	61	3	5		1	0	161	5	2
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	12	2	2	62	2	5	112	7	5	162	7	5
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	1/	5	2	6/	2	1	115	7	4	164	7	4
16316632113351633416316632116141661417326713117551677581326812118751683391136953119171693420527073120741707421137114121751715522337212122351727423547313123161733524727453124751745525737674125361753626337653126561761337141273717714287378741283117814293479151293217935305480 </td <td>14</td> <td>2</td> <td>5</td> <td>65</td> <td>2</td> <td>1</td> <td>114</td> <td>5</td> <td>- 5</td> <td>165</td> <td>3</td> <td>5</td>	14	2	5	65	2	1	114	5	- 5	165	3	5
1031003211014100141732671311755168339113695311917169342052707312074170742113711412175171552233721212235172742354731312316173352472745312475174552573765312656176132714127371771428737874128311781429347915129321793530548052130321807431357413353183763332837412932179353054805 <td>16</td> <td>3</td> <td>4</td> <td>66</td> <td>3</td> <td>2</td> <td>115</td> <td>1</td> <td>5</td> <td>166</td> <td>1</td> <td>4</td>	16	3	4	66	3	2	115	1	5	166	1	4
17 3 2 68 1 2 118 7 5 166 3 4 20 5 2 70 7 3 120 7 4 170 7 4 20 5 2 70 7 3 120 7 4 170 7 4 21 1 3 71 1 4 121 7 5 171 5 22 3 3 72 1 2 122 3 5 172 7 4 23 5 4 73 1 3 123 1 6 173 3 5 24 7 2 74 5 3 124 7 5 174 5 5 25 7 3 75 7 4 125 3 6 175 3 6 26 3 3 76 5 3 126 5 6 176 1 3 27 1 4 127 3 7 177 1 4 28 7 3 78 7 4 128 3 1 178 1 4 29 3 4 79 1 5 129 3 2 179 3 5 30 5 4 80 5 2 130 3 2 180 7 4 413 <	17	3	2	67	1	2	117	5	5	167	7	5
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	81	3	2	68	1	2	118	7	5	168	, 3	3
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	91	1	3	69	5	3	119	í	7	169	3	4
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	20	5	2	70	7	3	120	7	4	170	7	4
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	21	ĩ	3	71	1	4	121	7	5	171	5	5
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	22	3	3	72	1	2	122	3	5	172	7	4
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	23	5	4	73	1	3	123	1	6	173	3	5
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	24	7	2	74	5	3	124	7	5	174	5	5
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	25	7	3	75	7	4	125	3	6	175	3	6
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	26	3	3	76	5	3	126	5	6	176	1	3
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	27	1	4	77	1	4	127	3	7	177	1	4
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	28	7	3	78	7	4	128	3	1	178	1	4
30 5 4 80 5 2 130 3 2 180 7 4 31 3 5 81 5 3 131 1 3 181 3 5 32 3 1 82 5 3 132 1 2 182 1 5 33 3 2 83 7 4 133 5 3 183 7 6 34 3 2 84 3 3 134 7 3 184 1 4 35 1 3 85 7 4 135 1 4 185 1 5 36 1 2 86 5 4 136 1 2 186 5 5 37 5 3 87 3 5 137 1 3 187 7 6 38 7 3 88 1 3 138 5 3 188 1	29	3	4	79	1	5	129	3	2	179	3	5
31 3 5 81 5 3 131 1 3 181 3 5 32 3 1 82 5 3 132 1 2 182 1 5 33 3 2 83 7 4 133 5 3 183 7 6 34 3 2 84 3 3 134 7 3 184 1 4 35 1 3 85 7 4 135 1 4 185 1 5 36 1 2 86 5 4 136 1 2 186 5 5 37 5 3 87 3 5 137 1 3 187 7 6 38 7 3 88 1 3 138 5 3 188 1 5	30	5	4	80	5	2	130	3	2	180	7	4
32 3 1 82 5 3 132 1 2 182 1 5 33 3 2 83 7 4 133 5 3 183 7 6 34 3 2 84 3 3 134 7 3 184 1 4 35 1 3 85 7 4 135 1 4 185 1 5 36 1 2 86 5 4 136 1 2 186 5 5 37 5 3 87 3 5 137 1 3 187 7 6 38 7 3 88 1 3 138 5 3 188 1 5	31	3	5	81	5	3	131	1	3	181	3	5
33 3 2 83 7 4 133 5 3 183 7 6 34 3 2 84 3 3 134 7 3 184 1 4 35 1 3 85 7 4 135 1 4 185 1 5 36 1 2 86 5 4 136 1 2 186 5 5 37 5 3 87 3 5 137 1 3 187 7 6 38 7 3 88 1 3 138 5 3 188 1 5	32	3	1	82	5	3	132	1	2	182	1	5
34 3 2 84 3 3 134 7 3 184 1 4 35 1 3 85 7 4 135 1 4 185 1 5 36 1 2 86 5 4 136 1 2 186 5 5 37 5 3 87 3 5 137 1 3 187 7 6 38 7 3 88 1 3 138 5 3 188 1 5	33	3	2	83	7	4	133	5	3	183	7	6
35 1 3 85 7 4 135 1 4 185 1 5 36 1 2 86 5 4 136 1 2 186 5 5 37 5 3 87 3 5 137 1 3 187 7 6 38 7 3 88 1 3 138 5 3 188 1 5	34	3	2	84	3	.3	134	7	3	184	1	4
36 1 2 86 5 4 136 1 2 186 5 5 37 5 3 87 3 5 137 1 3 187 7 6 38 7 3 88 1 3 138 5 3 188 1 5	35	1	3	85	7	4	135	1	4	185	1	5
37 5 3 87 3 5 137 1 3 187 7 6 38 7 3 88 1 3 138 5 3 188 1 5	36	1	2	86	5	4	136	1	2	186	5	5
38 / 3 88 1 3 138 5 3 186 1 5	37	5	3	8/	3	5	13/	1	5	18/	1	5
	38	/	3	88	1	3	138	2	3	100	1	2
39 1 4 89 1 4 139 7 4 109 5 0	39	1	4	89	1	4	139	/	4	109	2	6
$40 \ 5 \ 2 \ 90 \ 5 \ 4 \ 140 \ 5 \ 5 \ 190 \ 5 \ 0 \ 5 \ 0 \ 5 \ 0 \ 5 \ 0 \ 5 \ 0 \ 5 \ 0 \ 0$	40	5	2	90	2	4	140	5	5	190	5	7
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	41	5	3	91	/	5	141	7	4	192	7	2
42 1 5 32 1 4 142 7 4 152 7 2	42	1	2	92	5	4 5	1/12	1	5	193	7	3
$43 \ 3 \ 4 \ 3 \ 5 \ 5 \ 144 \ 1 \ 2 \ 194 \ 7 \ 3$	45	1	4	95	3	5	144	ī	2	194	7	3
44 1 3 195 5 6 145 1 3 195 5 4	44	5	5	95	5	6	145	ĩ	3	195	5	4
46 3 4 96 7 2 146 1 3 196 5 3	45	2		96	7	2	146	1	3	196	5	3
$40 \ 5 \ 4 \ 50 \ 7 \ 2 \ 140 \ 1 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 5 \ 5 \ 100 \ 100\ \ 100 \ 100 \ 100 \ 100 \ 100 \ 100 \ 1$	40	5	4 5	07	7	∠ ૧	147	3	4	197	1	4
47 5 5 98 7 3 148 7 3 198 3 4	47 48	כ ד	2	97	7	3	148	7	3	198	3	4
49 7 3 99 5 4 149 3 4 199 5 5	49	7	3	99	, 5	4	149	3	4	199	5	5
50 7 3 100 5 3 150 1 4 200 5 3	FO	, 7	3	100	5	3	150	1	4	200	5	3

REPRESENTING $\binom{2n}{n}$ As a sum of squares

Theorem 3: $\binom{2n}{n} \neq a^2 + b^2 + c^2$ iff $t_2(n)$ is even and g(2n) = 7. Proof: The proof follows from (5), (6), Lemma 6, and Definition 4. Theorem 4: Let k be a nonnegative integer. Then

- (a) g(8k) = g(4k); (b) g(8k + 2) = g(4k + 1);
- (c) $g(8k + 4) \equiv 3g(4k + 2) \pmod{8}$; (d) g(8k + 6) = 8 g(4k + 3).

Proof of (a): By Definition 4 and Lemma 4, it suffices to show that

$$f\left(\frac{(8k)!}{(4k)!}\right) = 1 \text{ for all } k \ge 0.$$

We proceed by induction on k. The statement is trivially true for k = 0. Now

$$f\left(\frac{(8(k+1)!)}{(4(k+1)!)}\right) = f\left(\frac{(8k+8)!)}{(4k+4)!}\right) = f\left(\frac{(8k+8)!(4k)!(8k)!}{(8k)!(4k+4)!(4k)!}\right)$$
$$= f\left(\frac{(8k+8)!(4k)!}{(8k)!(4k+4)!}\right)$$

by induction hypothesis and Lemma 4. But

 $f\left(\frac{(8k+8)!(4k)!}{(8k)!(4k+4)!}\right)$ = $f\left(\frac{(8k+8)(8k+7)(8k+6)(8k+5)(8k+4)(8k+3)(8k+2)(8k+1)}{(4k+4)(4k+3)(4k+2)(4k+1)}\right)$ = $f(2^{4}(8k+7)(8k+5)(8k+3)(8k+1) = f(7 \cdot 5 \cdot 3 \cdot 1) = f(105) = 1.$

Parts (b), (c), and (d) may be proved in similar fashion.

Theorem 5: $g(2m) = \begin{cases} g(m) & \text{if } m \equiv 1 \pmod{4}, \\ 8 - g(m) & \text{if } m \equiv 3 \pmod{4}. \end{cases}$

Proof: The proof follows from Theorem 4.

Theorem 6: If either (i) $m \equiv 1 \pmod{4}$ and g(m) = 5, or (ii) $m \equiv -1 \pmod{4}$ and g(m) = 3, then g(2m) = 5 and g(4m) = 7.

Proof: The hypothesis and Theorem 5 imply q(2m) = 5. Now $m = 4p \pm 1$, so

$$g(4m) = g(4(4r \pm 1)) = g(8(2r) \pm 4) \equiv 3g(4(2r) \pm 2) \equiv 3g(2(4r \pm 1)),$$

 $3g(2m) \equiv 3 \cdot 5 \equiv 7 \pmod{8}$,

by Theorem 4(c). Therefore, g(4m) = 7.

Theorem 7: If *m* is odd and g(2m) = 5, then $g(2^km) = 7$ for all $k \ge 2$.

Proof: (Induction on k.) By Theorem 6, the statement is true for k = 2. If k > 2, then $g(2^{k}m) = g(8(2^{k-3}m)) = g(4(2^{k-3}m)) = g(2^{k-1}m) = 7$, by Theorem 4(a) and the induction hypothesis.

[Feb.

32

Theorem 8: S_3 is infinite, that is, there exist infinitely many n such that

 $\binom{2n}{n} = a^2 + b^2 + c^2.$

Proof: If $m \ge 2$, then $t_2(2^{2m-1}-1) = 2m - 1$, and $2^{2m-1} - 1 \ge 3$, so that Theorems 2 and 3 imply that $2^{2m-1} - 1$ belongs to S_3 .

Theorem 9: S_4 is infinite, that is, there exist infinitely many n such that

$$\binom{2n}{n} \neq a^2 + b^2 + c^2.$$

Proof: By Theorems 3, 6, and 7, it suffices to find an m such that (i) $t_2(m)$ is even, and either (ii) $m \equiv 1 \pmod{4}$ and g(m) = 5, or (iii) $m \equiv 3 \pmod{4}$ and g(m) = 3. Examining Table 1, we find the following such m < 200:

 $m \in \{3, 15, 43, 53, 63, 147, 153, 175, 189\}.$

Concluding Remarks: Let d_n be the asymptotic density of S_n , where $1 \le n \le 4$. Since $S_1 \cup S_2$ is finite, by Theorem 2, we have $d_1 = d_2 = 0$, so that $d_3 + d_4 = 1$. If *n* is a randomly chosen natural number, let *A* be the event that $t_2(n)$ is even; let *B* be the event that g(2n) = 7. It is easily seen that $\Pr(A) = \frac{1}{2}$. Now $d_4 = \Pr(n \in S_4) = \Pr(A \cap B) \le \Pr(A) = \frac{1}{2}$. Therefore, $d_3 \ge \frac{1}{2}$. Table 1 suggests that *A* and *B* are independent, and that $\Pr(B) = \frac{1}{4}$.

Conjecture: $d_4 = 1/8$, $d_3 = 7/8$.

REFERENCES

- 1. R. Breusch. "Zur Allgemeinerung des Bertrandschen Postulates, dass zwischen x und 2x stets Primzahlen liegen." Math. Zeit. 34 (1932):505-526.
- 2. David M. Burton. Elementary Number Theory. Boston: Allyn & Bacon, 1976.
- 3. P. Erdös. "On a Diophantine Equation." J. London Math. Soc. 26 (1951): 176-178.
- 4. G. H. Hardy & E. M. Wright. *The Theory of Numbers*. Oxford: Oxford University Press, 4th ed., 1960.
- 5. Stephen Wolfram. "Geometry of Binomial Coefficients." Amer. Math. Monthly 91 (1984):566-571.

1987]