ANOTHER FAMILY OF FIBONACCI-LIKE SEQUENCES

PETER R. J. ASVELD
Department of Computer Science, Twente University of Technology
P.O. Box 217, 7500 AE Enschede, The Netherlands

(Submitted March 1986)

In [1] we studied the class of recurrence relations

$$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} \alpha_j n^j \tag{1}$$

with $G_0 = G_1 = 1$. The main result of [1] consists of an expression for G_n in terms of the Fibonacci numbers F_n and F_{n-1}, and in the parameters $\alpha_0, \ldots, \alpha_n$.

The present note is devoted to the related family of recurrences that is obtained by replacing the (ordinary or power) polynomial in (1) by a factorial polynomial; viz.

$$H_n = H_{n-1} + H_{n-2} + \sum_{j=0}^{k} \gamma_j n^{(j)} \tag{2}$$

with $H_0 = H_1 = 1$, $n^{(j)} = n(n-1)(n-2) \ldots (n-j+1)$ for $j \geqslant 1$, and $n^{(0)} = 1$. The structure of this note resembles the one of [1] to a large extent.

As usual (cf. e.g., [2] and [4]) the solution $H_n^{(h)}$ of the homogeneous equation corresponding to (2) is

$$H_n^{(h)} = C_1\phi_1^n + C_2\phi_2^n$$

with $\phi_1 = \frac{1}{2}(1+\sqrt{5})$ and $\phi_2 = \frac{1}{2}(1-\sqrt{5})$.

Next we try as a particular solution

$$H_n^{(p)} = \sum_{i=0}^{k} B_i n^{(i)},$$

which yields

$$\sum_{i=0}^{k} B_i n^{(i)} - \sum_{i=0}^{k} B_i (n-1)^{(i)} - \sum_{i=0}^{k} B_i (n-2)^{(i)} - \sum_{i=0}^{k} \gamma_i n^{(i)} = 0.$$

In order to rewrite this equality, we need the following *Binomial Theorem for Factorial Polynomials*.

Lemma 1: $(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}.$

Proof (A. A. Jagers):

$$(x+y)^{(n)} t^{x+y} = t^n \frac{d^n t^{x+y}}{dt^n}$$

(Leibniz's formula)

$$= t^n \sum_{k=0}^{n} \binom{n}{k} x^{(k)} t^{x-k} y^{(n-k)} t^{y-n+k}.$$

Cancellation of t^{x+y} yields the desired equality. ■

Thus, we have

$$\sum_{i=0}^{k} B_i n^{(i)} - \sum_{\ell=0}^{k}\left(\sum_{i=0}^{i} B_i \binom{i}{\ell}\left((-1)^{(i-\ell)} + (-2)^{(i-\ell)}\right)n^{(\ell)}\right) - \sum_{i=0}^{k} \gamma_i n^{(i)} = 0;$$

hence, for each i $(0 \leqslant i \leqslant k)$,

$$B_i - \sum_{m=i}^{k} \delta_{im} B_m - \gamma_i = 0 \tag{3}$$

with, for $m \geqslant i$,

$$\delta_{im} = \binom{m}{i}((-1)^{(m-i)} + (-2)^{(m-i)}).$$

Since $(-x)^{(n)} = (-1)^n(x + n - 1)^{(n)}$ and $n^{(n)} = n!$, we have

$$\begin{aligned}\delta_{im} &= \binom{m}{i}(-1)^{m-i}((m - i)! + (m - i + 1)!)\\ &= \binom{m}{i}(-1)^{m-i}(m - i + 2)(m - i)!\\ &= (-1)^{m-i}(m - i + 2)m^{(m-i)}.\end{aligned}$$

From the family of recurrences (3), we can successively determine $B_k, \ldots, B_0$: the coefficient B_i is a linear combination of $\gamma_i, \ldots, \gamma_k$. Therefore, we set

$$B_i = -\sum_{j=i}^{k} b_{ij}\gamma_j$$

(cf. [1]) which yields, together with (3),

$$-\sum_{j=i}^{k} b_{ij}\gamma_j + \sum_{m=i}^{k} \delta_{im}\left(\sum_{\ell=m}^{k} b_{m\ell}\gamma_\ell\right) - \gamma_i = 0.$$

Thus, for $0 \leqslant i \leqslant j \leqslant k$, we have

$$\begin{aligned}b_{jj} &= 1\\ b_{ij} &= -\sum_{m=i+1}^{j} \delta_{im} b_{mj}, \text{ if } i < j.\end{aligned}$$

Hence, for the particular solution $H_n^{(p)}$ of (2), we obtain

$$H_n^{(p)} = -\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}\gamma_j n^{(i)} = -\sum_{j=0}^{k}\gamma_j\left(\sum_{i=0}^{j} b_{ij} n^{(i)}\right).$$

As in [1] the determination of C_1 and C_2 from $H_0 = H_1 = 1$ yields

$$H_n = (1 - H_0^{(p)})F_n + (-H_1^{(p)} + H_0^{(p)})F_{n-1} + H_n^{(p)}.$$

Therefore, we have

Proposition 2: The solution of (2) can be expressed as

$$H_n = (1 + M_k)F_n + \mu_k F_{n-1} - \sum_{j=0}^{k}\gamma_j \pi_j(n),$$

where M_k is a linear combination of $\gamma_0, \ldots, \gamma_k$, μ_k is a linear combination of $\gamma_1, \ldots, \gamma_k$, and for each j $(0 \leqslant j \leqslant k)$, $\pi_j(n)$ is a factorial polynomial of degree j:

$$M_k = \sum_{j=0}^{k} b_{0j}\gamma_j, \quad \mu_k = \sum_{j=1}^{k} b_{1j}\gamma_j, \quad \pi_j(n) = \sum_{i=0}^{j} b_{ij}n^{(i)}. \quad \blacksquare$$

Table 1

j	$\pi_j(n)$
0	1
1	$n^{(1)} + 3$
2	$n^{(2)} + 6n^{(1)} + 10$
3	$n^{(3)} + 9n^{(2)} + 30n^{(1)} + 48$
4	$n^{(4)} + 12n^{(3)} + 60n^{(2)} + 192n^{(1)} + 312$
5	$n^{(5)} + 15n^{(4)} + 100n^{(3)} + 480n^{(2)} + 1560n^{(1)} + 2520$
6	$n^{(6)} + 18n^{(5)} + 150n^{(4)} + 960n^{(3)} + 4680n^{(2)} + 15120n^{(1)} + 24480$
7	$n^{(7)} + 21n^{(6)} + 210n^{(5)} + 1680n^{(4)} + 10920n^{(3)} + 52920n^{(2)} + 171360n^{(1)} + 277200$
8	$n^{(8)} + 24n^{(7)} + 280n^{(6)} + 2688n^{(5)} + 21840n^{(4)} + 141120n^{(3)} + 685440n^{(2)} +$ $+ 2217600n^{(1)} + 3588480$
9	$n^{(9)} + 27n^{(8)} + 360n^{(7)} + 4032n^{(6)} + 39312n^{(5)} + 317520n^{(4)} + 2056320n^{(3)} +$ $+ 9979200n^{(2)} + 32296320n^{(1)} + 52254720$

Table 1 displays the factorial polynomials $\pi_j(n)$ for $j = 0, 1, \ldots, 9$.

The coefficients of $\gamma_0, \gamma_1, \gamma_2, \ldots$ in M_k and of $\gamma_1, \gamma_2, \ldots$ in μ_k are independent of k; cf. [1]. As k tends to infinity they give rise to two infinite sequences M and μ of natural numbers (not mentioned in [3]) of which the first few elements are

M: 1, 3, 10, 48, 312, 2520, 24480, 277200, 3588480, 52254720, ...

μ: 1, 6, 30, 192, 1560, 15120, 171360, 2217600, 322963 , ...

Contrary to the corresponding sequences Λ and λ in [1], M and μ obviously show more regularity. Formally, this is expressed in

Proposition 3: For each i and j with $0 \leqslant i \leqslant j \leqslant k$,

$$b_{jj} = 1$$
$$b_{ij} = j^{(j-i)}F_{j-i+2}, \text{ if } i < j.$$

Consequently,

$$M_k = \gamma_0 + \sum_{j=1}^{k} j!F_{j+2}\gamma_j \quad \text{and} \quad \mu_k = \gamma_1 + \sum_{j=2}^{k} j!F_{j+1}\gamma_j.$$

Proof: The argument proceeds by induction on $j - i$.

Initial step $(j - i = 1)$: $b_{j-1,j} = -\delta_{j-1,j}b_{jj} = -(-1)^1 \cdot 3j \cdot 1 = j^{(1)}F_3$.

Induction hypothesis: For all m with $i < m < j$, $b_{mj} = j^{(j-m)}F_{j-m+2}$.

Induction step: $b_{ij} = -\sum_{m=i+1}^{j} \delta_{im} b_{mj} = -\delta_{ij} b_{jj} - \sum_{m=i+1}^{j-1} \delta_{im} b_{mj}$

$$= (-1)^{j-i+1}(j - i + 2)j^{(j-i)} + \sum_{m=i+1}^{j-1} (-1)^{m-i+1}(m - i + 2)m^{(m-i)} b_{mj}.$$

From the induction hypothesis, it follows that

$$b_{ij} = j^{(j-i)}\left((-1)^{j-i+1}(j - i + 2) + \sum_{m=i+1}^{j-1} (-1)^{m-i+1}(m - i + 2)F_{j-m+2}\right).$$

As $F_0 = F_1 = 1$, we may replace $j - i + 2$ by $F_0 + (j - i + 1)F_1$. Adding

$$j^{(j-i)}\Bigg((-1)^{j-i}(F_0 + F_1 - F_2) + \sum_{m=i+1}^{j-1} (-1)^{m-i+1}(m - i + 1)(F_{j-m} + F_{j-m+1} - F_{j-m+2})\Bigg) = 0$$

yields, after rearranging,

$$b_{ij} = j^{(j-i)}(F_{j-i} + F_{j-i+1}) = j^{(j-i)}F_{j-i+2},$$

which completes the induction. ■

Clearly, Proposition 3 provides a different way of computing the coefficients a_{ij} (and hence the elements of the sequences Λ and λ) from [1]; viz. by

$$a_{ij} = \sum_{m=i}^{j} s(i, m) \left(\sum_{\ell=m}^{j} b_{m\ell} S(\ell, j) \right) \qquad (i \leqslant j),$$

where $s(i, m)$ and $S(\ell, j)$ are the Stirling numbers of the first and second kind, respectively.

ACKNOWLEDGMENTS

For some useful discussions, I am indebted to Frits Göbel and particularly to Bert Jagers who brought factorial polynomials to my notice and provided the proof of Lemma 1.

REFERENCES

1. P. R. J. Asveld. "A Family of Fibonacci-Like Sequences." *The Fibonacci Quarterly* 25, no. 1 (1987):81-83.
2. C. L. Liu. *Introduction to Combinatorial Mathematics*. New York: McGraw-Hill, 1968.
3. N. J. A. Sloane. *A Handbook of Integer Sequences*. New York: Academic Press, 1973.
4. N. N. Vorobyov. *The Fibonacci Numbers*. Boston: Heath, 1963.

◆◇◆◇◆