ON CERTAIN SEMI-PERFECT CUBOIDS

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1. The classical cuboid is a rectangular block with integral edges and face diagonals. If we consider the internal diagonal as well, then there are seven lengths in all. It is known [3] that any six of the seven lengths can be integral. We can call such cuboids semi-perfect. Semi-perfect cuboids fall into three categories such that there is no integral specification for:

(1) the internal diagonal,
(2) one face diagonal,
(3) one edge.

If all seven lengths were integral, then we would have what is known as a perfect cuboid. No such perfect cuboids are known; indeed, their existence is a classical open question. It is known [3] that there are an infinity of semi-perfect cuboids in all categories, as certain parametric solutions are known. Unfortunately, none of these solutions is complete. Clearly, if perfect cuboids exist, they must fall into all three categories and so the complete determination of all semi-perfect cuboids in any one category would reduce the problem of perfect cuboids to the consideration of the seventh nonspecified length. It has been shown that some of these partial parametric solutions cannot be perfect (see [2], [3], and [4]). In this paper we shall determine a two-parameter solution for category (3) which is the generalization of a solution first given by Bromhead [1], and then show in a simple manner that this too can never give a perfect cuboid.

2. It is instructive first to consider the smallest real solutions (with $c > 0$) in category (3). If we measure the size of the cuboid by the length of the internal diagonal d (say) with edges a, b, and $\sqrt{c}$, then Leech [3] has given the smallest solutions. The first four being

a	b	c	d
520	576	618849	1105
1800	1443	461776	2405
1480	969	6761664	3145
124	957	13852800	3845

where $a^2 + b^2$, $a^2 + c$, $b^2 + c$, and $a^2 + b^2 + c$ are all square. If $\sqrt{c}$ were itself integral, then of course the cuboid would be perfect.

3. Following Bromhead's solution [1], we have $a^2 + b^2$ square and, hence,

$$a = k(2uv) \quad \text{and} \quad b = k(u^2 - v^2).$$

If we write $a^2 + c = p^2$ and $b^2 + c = q^2$, then $p^2 + b^2 = q^2 + a^2$, and each is a square. Therefore, $p = k_1(2u_1v_1)$ or $k_1(u_1^2 - v_1^2)$ with $k_1(u_1^2 - v_1^2) = k(u^2 - v^2)$ or $k_1(2u_1v_1) = k(u^2 - v^2)$. Similarly, $q = k_2(u_2^2 - v_2^2)$ or $k_2(2u_2v_2)$; hence,

$$k_2(2u_2v_2) = k(2uv) \quad \text{or} \quad k_2(u_2^2 - v_2^2) = k(u^2 - v^2).$$

Finally, $k_1(u_1^2 + v_1^2) = k_2(u_2^2 + v_2^2)$ in all cases.

Since we need only consider cuboids with $(a^2, b^2, c) = 1$, we can reduce the problem to solving the systems:

3.1 $$\begin{array}{lll} (1) & k_1(u_1^2 + v_1^2) = k_2(u_2^2 + v_2^2) & \text{or} \quad (2) & k_1(u_1^2 + v_1^2) = k_2(u_2^2 + v_2^2) \\ & k_1(u_1^2 - v_1^2) = k(u^2 - v^2) & & k_1(2u_1v_1) = k(u^2 - v^2) \\ & k_2(2u_2v_2) = k(2uv) & & k_2(2u_2v_2) = k(2uv) \end{array}$$

in integers. Thus, we can say that all primitive semi-perfect cuboids in category (3) must satisfy either system (1) or system (2). Of the four "smallest" real solutions listed above the smallest satisfies system (2) and the next three satisfy system (1). Bromhead's one-parameter solution satisfies system (1) when $k = k_1 = k_2$, and the smallest solution with this condition is the fourth.

4. We shall now determine a two-parameter solution of system (1) when $k = k_1 = k_2$. We have:

4.1 $$u_1^2 + v_1^2 = u_2^2 + v_2^2;$$

4.2 $$u_1^2 - v_1^2 = u^2 - v^2;$$

4.3 $$u_2v_2 = uv.$$

The general solution of 4.1 is

4.4 $$(mp + nq)^2 + (mq - np)^2 = (mq + np)^2 + (mp - nq)^2.$$

Writing 4.2 as $u_1^2 + v^2 = u^2 + v_1^2$, its solution is:

4.5 $$(m_1p_1 + n_1q_1)^2 + (m_1q_1 - n_1p_1)^2 = (m_1p_1 - n_1q_1)^2 + (m_1q_1 + n_1p_1)^2.$$

Putting $$mp + nq = m_1p_1 + n_1q_1$$
and $$mq - np = m_1q_1 + n_1p_1,$$
a rational solution is given by:

4.6 $$p_1 = q,\ q_1 = p,\ m_1 = \frac{n(q^2 + p^2)}{q^2 - p^2},\ n_1 = \frac{m(q^2 - p^2) - 2npq}{q^2 - p^2}$$

Therefore,

4.7 $$u = (nq(q^2 + 3p^2) - mp(q^2 - p^2))/q^2 - p^2,$$

4.8 $$v = (np(3q^2 + p^2) - mq(q^2 - p^2))/q^2 - p^2.$$

Finally, we require, from 4.3, that

4.9 $$(mq + np)(mp - nq) = \left(mp - \frac{nq(q^2 + 3p^2)}{q^2 - p^2}\right)\left(mq - \frac{np(3q^2 + p^2)}{q^2 - p^2}\right)$$

Let $n = \lambda(q^2 - p^2)$, then

4.10 $$(mq + \lambda p(q^2 - p^2))(mp - \lambda q(q^2 - p^2))$$
$$= (mp - \lambda q(q^2 + 3p^2))(mq - \lambda p(3q^2 + p^2)).$$

Multiplying in (4.10) and simplifying, we have,

$$2mpq = \lambda(q^2 + p^2)^2;$$

therefore,

$$m = \frac{\lambda(q^2 + p^2)^2}{2pq}.$$

Let $\lambda = 2pq$, then

$$n = 2pq(q^2 - p^2) \quad \text{and} \quad m = (q^2 + p^2)^2.$$

Hence, we have a solution where:

4.11 $$\begin{aligned} u_1 &= p(q^2 + p^2)^2 + 2pq^2(q^2 - p^2) = p(p^4 + 3q^4); \\ v_1 &= q(q^2 + p^2)^2 - 2p^2q(q^2 - p^2) = q(3p^4 + q^4); \\ u_2 &= q(q^2 + p^2)^2 + 2p^2q(q^2 - p^2) = q(q^4 + 4q^2p^2 - p^4); \\ v_2 &= p(q^2 + p^2)^2 - 2pq^2(q^2 - p^2) = p(p^4 + 4p^2q^2 - q^4); \\ u &= 2pq^2(q^2 + 3p^2) - p(q^2 + p^2)^2 = p(q^4 + 4p^2q^2 - p^4); \\ v &= 2p^2q(3q^2 + p^2) - q(q^2 + p^2)^2 = q(p^4 + 4p^2q^2 - q^4). \end{aligned}$$

This gives the solution:

4.12 $$\begin{aligned} a &= 2pq(p^4 + 4p^2q^2 - q^4)(q^4 + 4p^2q^2 - p^4); \\ b &= p^2(q^4 + 4p^2q^2 - p^4)^2 - q^2(p^4 + 4p^2q^2 - q^4)^2; \\ c &= 32p^2q^2(p^4 - q^4)^2(p^8 + 14p^4q^4 + q^8). \end{aligned}$$

Bromhead's solution corresponds to $p = t + 1$, $q = t$. We need only consider values of p and q such that $(p, q) = 1$. If $p = 2$ and $q = 1$, we have:

$$a = 124; \quad b = 957; \quad c = 13852800.$$

The signs of a and b are, of course, irrelevant and so we will always take the absolute value.

5. We know that if c itself is square, then the cuboid will be perfect. Looking at the form given for c, this requires that $p^8 + 14p^4q^4 + q^8$ is twice a square. We shall now prove that this is not possible. Set

5.1 $\quad p^8 + 14p^4q^4 + q^8 = 2w^2$, where $(p, q) = 1$,

then p and q must both be odd, and

5.2 $\quad (p^4 - q^4)^2 + (4p^2q^2)^2 = 2w^2.$

The general solution of 5.2 is known to be

5.3 $\quad p^4 - q^4 = k(m^2 - 2mn - n^2) \quad \text{or} \quad k(m^2 + 2mn - n^2),$

5.4 $\quad 4p^2q^2 = k(m^2 + 2mn - n^2) \quad \text{or} \quad k(m^2 - 2mn - n^2)$

$$w = k(m^2 + n^2).$$

From 5.3 and 5.4,

$$p^4 - q^4 + 4p^2q^2 = 2k(m^2 - n^2).$$

If m and n have the same parity, then $8 | 2k(m^2 - n^2)$. However, $8 \nmid p^4 - q^4 + 4p^2q^2$ since $8 | p^4 - q^4$ but not $4p^2q^2$. Therefore, m and n must have opposite parities, in which case $m^2 \pm 2mn - n^2$ is odd. Hence, from 5.3, we have that $8 | k$. From 5.4, it follows that $8 | 4p^2q^2$, which is impossible because p and q are both odd. It also follows that c can never be square, so the semi-perfect cuboids generated by 4.12 can never be perfect.

REFERENCES

1. T. Bromhead. "On Square Sums of Squares." *Math. Gazette 44* (1960):219-20.
2. E. Z. Chein. "On the Derived Cuboid of an Eulerian Triple." *Can. Math. Bull. 20* (1977):509-10.
3. J. Leech. "The Rational Cuboid Revisited." *Amer. Math. Monthly 84* (1977): 518-33.
4. W. G. Spohn. "On the Integral Cuboid." *Amer. Math. Monthly 79* (1972):57-59.

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