FIBONACCI AND LUCAS CURVES

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(Submitted February 1986)

1. INTRODUCTION

Define the recurrence-generated sequence $\{H_n\}$ for integers n by

$$H_{n+2} = H_{n+1} + H_n, \quad H_0 = 2b, \quad H_1 = a + b \quad (n \geqslant 0) \tag{1.1}$$

where a and b are arbitrary but are generally considered to be integers. Negative subscripts of H can be included in an extended definition if necessary.

Using [2], equation (δ), we have, for the Binet form of this generalized sequence, *mutatis mutandis*,

$$H_n = \frac{A\alpha^n - B\beta^n}{\sqrt{5}} \tag{1.2}$$

where

$$\begin{cases} \alpha = \dfrac{1 + \sqrt{5}}{2} \\ \beta = \dfrac{1 - \sqrt{5}}{2} = -1/\alpha \end{cases} \tag{1.3}$$

are the roots of

$$\lambda^2 - \lambda - 1 = 0 \tag{1.4}$$

and

$$\begin{cases} A = a + b\sqrt{5} \\ B = a - b\sqrt{5} \end{cases} \tag{1.5}$$

From (1.2), it follows readily that

$$H_n = aF_n + bL_n \tag{1.6}$$

where

$$F_n = (\alpha^n - \beta^n)/\sqrt{5} \tag{1.7}$$

and

$$L_n = \alpha^n + \beta^n \tag{1.8}$$

are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively, occurring in (1.1), (1.2), and (1.6) when:

$$a = 1, \quad b = 0 \quad \text{for } F_n;$$
$$a = 0, \quad b = 1 \quad \text{for } L_n.$$

The explicit expressions (1.7) and (1.8) are the *Binet forms* of F_n and L_n.

Following an idea of Wilson [5], we set

$$x = \{A\alpha^{2n} + B\cos(n-1)\pi\}/\sqrt{5}\alpha^n \tag{1.9}$$

and

$$y = B\sin(n-1)\pi/\sqrt{5}\alpha^n \tag{1.10}$$

which we now regard as Cartesian coordinates in a plane (though Wilson [6] expressed his notion in terms of polar coordinates).

Certain geometrical features relating to circles and rectangular hyperbolas were shown [3] to be consequences of (1.9) and (1.10). These features were extended to Pell numbers and Pell-Lucas numbers in [4].

Here we examine (1.9) and (1.10) in a rather different geometrical context.

2. GENERALIZED BINET FORMS

First, we generalize (1.9) and (1.10) from an integer exponent n to a real exponent θ:

$$x = \{A\alpha^{2\theta} + B\cos(\theta-1)\pi\}/\sqrt{5}\alpha^\theta; \tag{2.1}$$

$$y = B\sin(\theta-1)\pi/\sqrt{5}\alpha^\theta. \tag{2.2}$$

Expanding the trigonometrical components of (2.1) and (2.2), we find

$$x = \{A\alpha^\theta - B\alpha^{-\theta}\cos\theta\pi\}/\sqrt{5} \tag{2.3}$$

and

$$y = -B\alpha^{-\theta}\sin\theta\pi/\sqrt{5}. \tag{2.4}$$

We will be particularly interested in the Fibonacci and Lucas aspects of (2.3). For the Fibonacci case $a = 1$, $b = 0$, so $A = B = 1$, and (2.3) becomes, with (1.3),

$$x = \frac{\alpha^\theta - \alpha^{-\theta}\cos\theta\pi}{\sqrt{5}} = \{\alpha^\theta - (-1)^\theta\beta^\theta\cos\theta\pi\}/\sqrt{5} \tag{2.5}$$

while for the Lucas case $a = 0$, $b = 1$, so $A = -B = \sqrt{5}$, and (2.3) reduces to

$$x = \alpha^\theta + (-1)^\theta\beta^\theta\cos\theta\pi. \tag{2.6}$$

When θ is an integer n, (2.5) and (2.6) simplify to the Binet forms (1.7) and (1.8), respectively. Therefore, we are justified in referring to (2.5) and (2.6) as the *generalized Binet forms* of F_n and L_n, i.e., the Binet forms of F_θ and L_θ.

It is the object of this paper to consider, *inter alia*, the locus generated by the parametric equations (2.3) and (2.4). Efforts to express the equation of this locus in Cartesian form, i.e., to eliminate the parameter θ, have not met with success.

From (2.4) we have

$$\frac{dy}{d\theta} = \frac{B\alpha^{-\theta}}{\sqrt{5}}(\log \alpha \sin \theta\pi - \pi \cos \theta\pi) \tag{2.7}$$

while from (2.3)

$$\frac{dx}{d\theta} = \frac{\alpha^{-\theta}}{\sqrt{5}}\{A\alpha^{2\theta} \log \alpha + B(\log \alpha \cos \theta\pi + \pi \sin \theta\pi)\} \tag{2.8}$$

whence

$$\frac{dy}{dx} = \frac{B(\log \alpha \sin \theta\pi - \pi \cos \theta\pi)}{A\alpha^{2\theta} \log \alpha + B(\log \alpha \cos \theta\pi + \pi \sin \theta\pi)} = 0 \tag{2.9}$$

when

$$\tan \theta\pi = \frac{\pi}{\log \alpha} \quad (\doteqdot 6.53 \text{ to two decimal places}) \tag{2.10}$$

yielding

$$\theta\pi \doteqdot 81°18' \text{ from tables,} \tag{2.11}$$

that is,

$$\theta \doteqdot 0.45 \quad (\doteqdot 26° \text{ in degree measure}). \tag{2.12}$$

Thus, the stationary points on the curve occur when

$$\tan(\theta - m)\pi = \frac{\pi}{\log \alpha} \quad (m \text{ an integer}), \tag{2.13}$$

that is,

$$\theta = \frac{1}{\pi} \tan^{-1}\left(\frac{\pi}{\log \alpha}\right) + m. \tag{2.14}$$

The nature of these stationary points, i.e., whether they yield maxima or minima, can be determined by the usual elementary methods.

Next, we discover the locus of the stationary points.

Write

$$\sin(\theta - m)\pi = k\pi \quad \text{i.e., } \sin \theta\pi = \pm k\pi \tag{2.15}$$

and

$$\cos(\theta - m)\pi = k \log \alpha \quad \text{i.e., } \cos \theta\pi = \pm k \log \alpha, \tag{2.16}$$

where

$$k = (\pi^2 + \log^2\alpha)^{-1/2} \qquad (\doteqdot 3.2). \tag{2.17}$$

Because $\sin\theta\pi$ and $\cos\theta\pi$ (and therefore θ) now have specified numerical values for the stationary points, we can eliminate α^θ from (2.3) and (2.4).

Substitute from (2.15) and (2.16) in (2.3) and (2.4) to obtain

$$\sqrt{5}x \cdot \mp \frac{Bk\pi}{\sqrt{5}y} = \frac{AB^2k^2\pi^2}{5y^2} \mp Bk \log\alpha$$

$$y^2 - \frac{\pi}{\log\alpha}xy = \frac{\pm ABk\pi^2}{5\log\alpha} \tag{2.18}$$

i.e., the branch in the first quadrant of the hyperbola,

$$y^2 - \frac{\pi}{\log\alpha}xy = \frac{ABk\pi^2}{5\log\alpha}, \tag{2.19}$$

and the branch in the fourth quadrant of the conjugate hyperbola,

$$y^2 - \frac{\pi}{\log\alpha}xy = -\frac{ABk\pi^2}{5\log\alpha}. \tag{2.20}$$

Common asymptotes of these two hyperbolas are

$$y = 0, \qquad y = \frac{\pi}{\log\alpha}x. \tag{2.21}$$

The oblique asymptote $y = \frac{\pi}{\log\alpha}x$ has gradient 81°18′ (approx.) by (2.10) and (2.11).

Of course, there are infinitely many points on (2.18) which do not satisfy (2.10), i.e., which are not stationary points. Therefore, the loci (2.18) are lacunary.

Inflections on the parametric curve (2.1) and (2.2) are given by the vanishing of $\frac{d^2y}{dx^2}$. Differentiating (2.9) a second time, we get

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx} \tag{2.22}$$

$$= \frac{[A\alpha^{2\theta}\log\alpha(3\pi\log\alpha\cos\theta\pi + (\pi^2 - 2\log^2\alpha)\sin\theta\pi) + B\pi k^2]\sqrt{5}\alpha^\theta}{\{A\alpha^{2\theta}\log\alpha + B(\log\alpha\cos\theta\pi + \pi\sin\theta\pi)\}^3}$$

after some simplification.

Inflections are then given by those values of θ for which

$$A\alpha^{2\theta}\log\alpha(3\pi\log\alpha\cos\theta\pi + (\pi^2 - 2\log^2\alpha)\sin\theta\pi) + B\pi k^2 = 0. \tag{2.23}$$

To test for maxima and minima, use (2.15)-(2.17), keeping in mind that

$$\pi\cos\theta\pi = \log\alpha\sin\theta\pi.$$

Then, at the stationary points (letting the variable θ be replaced by constants Θ), we find that the left-hand side of (2.23) is, after tidying up,

$$k^2\pi\{A\alpha^{2\Theta}\log\alpha. \pm k^{-3} + B\} \tag{2.24}$$

which becomes

$$k^2\pi\{\pm k^{-3}\alpha^{2\Theta}\log\alpha + 1\} \tag{2.25}$$

in the Fibonacci case, and

$$\frac{k^2\pi}{\sqrt{5}}\{\pm k^{-3}\alpha^{2\Theta}\log\alpha - 1\} \tag{2.26}$$

in the Lucas case.

If the numerical values of Θ are known, the nature of the turning points may be determined from (2.25) and (2.26). Note that $k^{-3}\alpha^{\Theta}\log\alpha$ is always positive.

No obviously derived differential equation satisfies (3.3) and (3.4) for the curve.

Finally, if we rewrite (2.3) and (2.4) as

$$x(\theta) = (A\alpha^{\theta} + (-1)^{\theta-1}B\beta^{\theta}\cos\theta\pi)/\sqrt{5} \tag*{(2.3)$'$}$$

and

$$y(\theta) = c(-1)^{\theta-1}\beta^{\theta}\sin\pi \tag*{(2.4)$'$}$$

(on putting $c = B/\sqrt{5}$ temporarily), we can see from the tables that the recurrence relation (1.1) is, in effect, satisfied as

$$x(\theta) = x(\theta - 1) + x(\theta - 2) \tag*{(2.3)$''$}$$

and

$$y(\theta) = y(\theta - 1) + y(\theta - 2). \tag*{(2.4)$''$}$$

The proofs follow. We have

$$\begin{aligned}
x(\theta - 1) &= (A\alpha^{\theta-1} + (-1)^{\theta-2}B\beta^{\theta-1}\cos(\theta - 1)\pi)/\sqrt{5}\\
&= (A\alpha^{\theta-1} + (-1)^{\theta-1}B\beta^{\theta-1}\cos\theta\pi)/\sqrt{5}\\
x(\theta - 2) &= (A\alpha^{\theta-2} + (-1)^{\theta-3}B\beta^{\theta-2}\cos(\theta - 2)\pi)/\sqrt{5}\\
&= (A\alpha^{\theta-2} + (-1)^{\theta-1}B\beta^{\theta-2}\cos\theta\pi)/\sqrt{5}\\
x(\theta - 1) + x(\theta - 2) &= (A\alpha^{\theta-2}(\alpha + 1) + (-1)^{\theta-1}B\beta^{\theta-2}(\beta + 1)\cos\theta\pi)/\sqrt{5}\\
&= (A\alpha^{\theta} + (-1)^{\theta-1}B\beta^{\theta}\cos\theta\pi)/\sqrt{5} = x(\theta)
\end{aligned}$$

as required, since α, β satisfy (1.4).

Similarly,

$$\begin{aligned}
y(\theta - 1) &= c(-1)^{\theta-2}\beta^{\theta-1}\sin(\theta - 1)\pi = c(-1)^{\theta-1}\beta^{\theta-1}\sin\theta\pi\\
y(\theta - 2) &= c(-1)^{\theta-3}\beta^{\theta-2}\sin(\theta - 2)\pi = c(-1)^{\theta-1}\beta^{\theta-2}\sin\theta\pi
\end{aligned}$$

and $$\begin{aligned} y(\theta - 1) + y(\theta - 2) &= c(-1)^{\theta-1}\beta^{\theta-2}(\beta + 1)\sin\theta\pi \\ &= c(-1)^{\theta-1}\beta^{\theta}\sin\theta\pi \qquad \text{since } \beta \text{ satisfies (1.4)} \\ &= y(\theta). \end{aligned}$$

Thus, it has been demonstrated that the parametric forms (2.3)″ and (2.4)″ do indeed satisfy recurrence relation (1.1).

We need this assurance to preserve the continuity of our curves in Figures 1, 2, and 3, which we now examine.

3. THE FIBONACCI CURVE

Table 1 sets out the values of x in (2.5), and y in (2.2) where $B = 1$, for the Fibonacci case $a = 1$, $b = 0$, when we proceed to increase θ by multiples of 0.2.

Table 1. The Fibonacci Curve

θ	x	y	θ	x	y
1	1	0			
1.2	0.999799314	0.14755316	6	8.000000000	0
1.4	0.947653586	0.216839615	6.2	8.817334649	-0.013304890
1.6	0.901827097	0.196943249	6.4	9.721923304	-0.019552416
1.8	0.911232402	0.110549283	6.6	10.71685400	-9.96822E-03
2	1	0	6.8	11.80690074	-9.96822E-03
2.2	1.163587341	-0.091192868	7	13.00000000	0
2.4	1.375792509	-0.134014252	7.2	14.3076953	8.22286E-03
2.6	1.602274541	-0.121717622	7.4	15.744608	0.012084058
2.8	1.814640707	-0.068323214	7.6	17.32733182	0.010975271
3	2.000000000	0	7.8	19.07328767	6.16070E-03
3.2	2.163386655	0.056360292	8	21.00000000	0
3.4	2.323446095	0.082825363	8.2	23.12502995	-5.08200E-03
3.6	2.504101639	0.075225627	8.4	25.4665313	-7.46836E-03
3.8	2.725873109	0.042226069	8.6	28.04418582	-6.78309E-03
4	3.000000000	0	8.8	30.8801884	-3.80752E-03
4.2	3.326973997	-0.034832576	9	34.00000000	0
4.4	3.699238605	-0.051188889	9.2	37.43272525	3.14085E-03
4.6	4.10637618	-0.046491995	9.4	41.21113931	4.611570E-03
4.8	4.540513816	-0.026097146	9.6	45.37151764	4.19218E-03
5	5.000000000	0	9.8	49.953447608	2.35318E-03
5.2	5.490360652	0.021527716	10	55.00000000	0
5.4	6.022684699	0.031636473			
5.6	6.610477819	0.028733633			
5.8	7.266386925	0.016128923			

Figure 1 shows the computer-drawn graph corresponding to the data in Table 1. We may call it the *Fibonacci curve*.

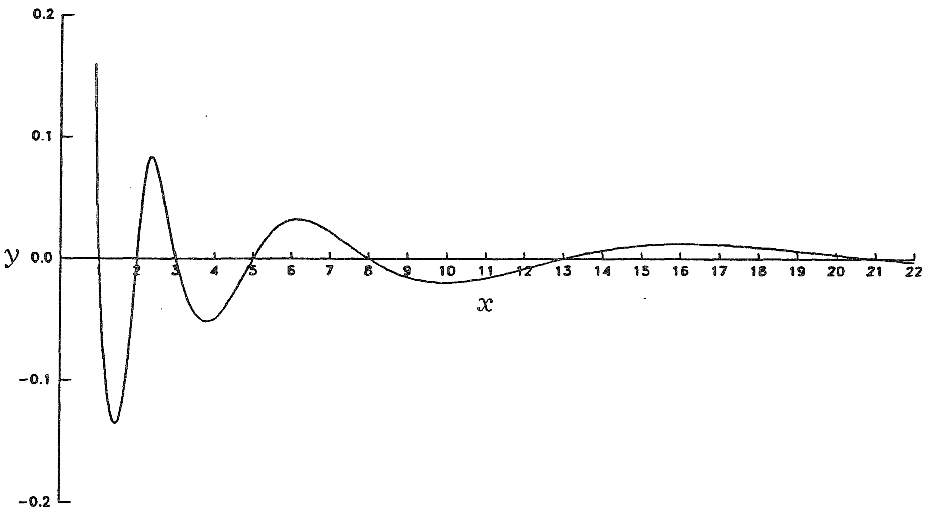

Figure 1. The Fibonacci Curve

Using (2.19) and (2.20) with $A = B = 1$ for the Fibonacci curve, we see that the locus of the stationary points is the appropriate branches of the hyperbolas

$$y^2 - \frac{\pi}{\log \alpha} xy = \pm \frac{k\pi^2}{5 \log \alpha}$$

From the observed stationary points on the plotted curve, one can visualize the need for a slight deviation (about 8.3°) from $x = 0$ of the "vertical" asymptote [refer to (2.11) and (2.21)]. The stationary points of the Fibonacci curve approach $y = 0$ asymptotically at a very quick rate (of necessity, since, in (2.2), $\alpha^\theta \to \infty$ rather rapidly as $\theta \to \infty$).

It is interesting to compare details of our Table 1 with similar figures given by Halsey [1]. See Table 2, in which the numbers in the first column for F_n are Halsey's and those in the second column for F_n are ours (to the same number of decimal places).

Starting from a quantity $n \not\Delta^m$ (read "*n delta-slash m*") which he defined for integers $m, n \geqslant 1$ and using the Pascal triangle generation of Fibonacci numbers (the elements of the Pascal triangle being expressed in terms of $n \not\Delta^m$ for various n and m), Halsey [1] established the following nice results:

$$F_n = \sum_{k=0}^{m} (n - 2k) \not\Delta^k \quad \left(\frac{n}{2} - 1 \leqslant m \leqslant \frac{n}{2}\right); \tag{3.1}$$

$$n \not\Delta^m = \binom{n + m - 1}{m}; \tag{3.2}$$

$$n \not\Delta^m = \left[(n + m)\int_0^1 x^{n-1}(1 - x)^m dx\right]^{-1}; \tag{3.3}$$

$$F_\theta = \sum_{k=0}^{m}\left[(\theta - k)\int_0^1 x^{\theta-2k-1}(1-x)^k dx\right]^{-1} \quad \left(\frac{\theta}{2} - 1 \leqslant m \leqslant \frac{\theta}{2}\right). \tag{3.4}$$

where θ is real.

Table 2

θ	F_θ	F_θ
2	1	1
2.2	1.2	1.2
2.4	1.4	1.4
2.6	1.6	1.6
2.8	1.8	1.8
3	2	2
3.2	2.2	2.2
3.4	2.4	2.3
3.6	2.6	2.5
3.8	2.8	2.7
4	3	3
4.2	3.32	3.33
4.4	3.68	3.70
4.6	4.08	4.11
4.8	4.52	4.54
5	5	5
5.2	5.52	5.49
5.4	6.08	6.02
5.6	6.68	6.61
5.8	7.32	7.27
6	8	8

To obtain the definite integral expressions, Halsey had recourse to basic properties of Beta functions and Gamma functions. It might be noted, as Halsey observed, that the Gamma function "extends the concept of factorials to numbers that are not integers," e.g., $\left(\frac{1}{2}\right)! = \sqrt{\pi}/2$. In this spirit, he extended the theory of Fibonacci numbers to noninteger values.

4. THE LUCAS CURVE

Table 3 lists the values of x in (2.5), and y in (2.2) where $B = -\sqrt{5}$, for the Lucas case $a = 0$, $b = 1$, when we increase θ by multiples of 0.2.

Figure 2 shows the computer-drawn graph corresponding to the data in Table 3. We may call it the *Lucas curve*.

As in the case of the Fibonacci curve, the locus of the stationary points on the Lucas curve, for which $A = -B = \sqrt{5}$, is the appropriate branches of the hyperbolas

$$y^2 - \frac{\pi}{\log \alpha} xy = \pm \frac{k\pi^2}{\log \alpha}.$$

Table 3. The Lucas Curve

θ	x	y	θ	x	y
1	0	0	6	18.00000002	0
1.2	1.327375368	-0.329938896	6.2	19.79805597	0.029750572
1.4	1.803931433	-0.484868119	6.4	21.76729273	0.043720531
1.6	2.302721986	-0.440378493	6.6	23.93780966	0.039708904
1.8	2.718049012	-0.247195712	6.8	26.33967462	0.022289623
2	3	0	7	29.00000003	0
2.2	3.16318597	0.203913452	7.2	31.94236463	-0.018386864
2.4	3.271099682	0.299664977	7.4	35.18845465	-0.027020774
2.6	3.405928737	0.272168877	7.6	38.76103987	-0.024541452
2.8	3.637105513	0.152775352	7.8	42.6870892	-0.013775745
3	4.000000002	0	8	47.00000006	0
3.2	4.49056134	-0.126025444	8.2	51.74042062	0.011363707
3.4	5.075031117	-0.185203141	8.4	56.95574739	0.016699757
3.6	5.708650725	-0.168209616	8.6	62.69884954	0.015167452
3.8	6.355154527	-0.094420360	8.8	69.02676384	8.51388E-03
4	7.000000004	0	9	76.0000001	0
4.2	7.653747312	0.077888006	9.2	83.68278528	-7.02316E-03
4.4	8.3461308	0.114461836	9.4	92.14420207	-0.010321017
4.6	9.114579464	0.103959260	9.6	101.4598894	-9.37400E-03
4.8	9.992260042	0.058354992	9.8	111.713853	-5.26187E-03
5	11	0	10	123.0000002	0
5.2	12.14430866	-0.048137436			
5.4	13.42116192	-0.070741305			
5.6	14.82323019	-0.064250356			
5.8	16.34641457	-0.036065368			

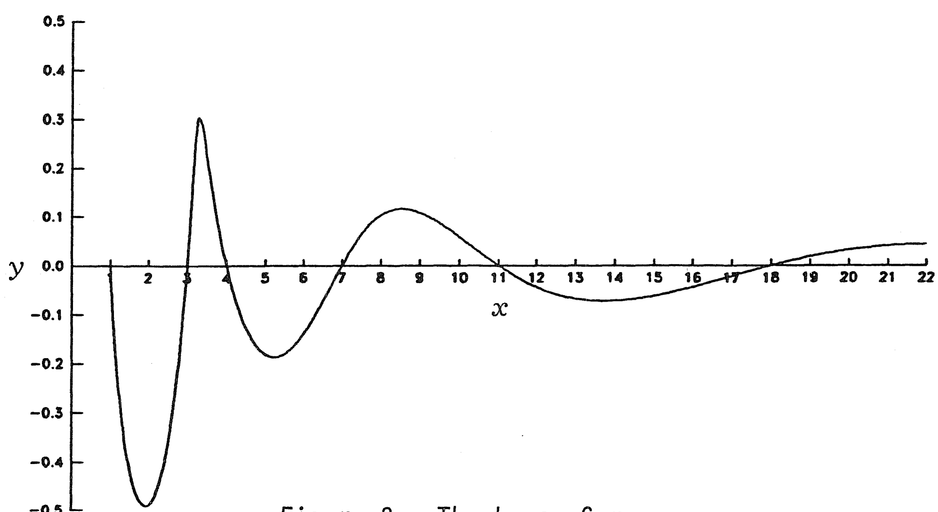

Figure 2. The Lucas Curve

Again, for the Lucas curve, the skewness (obliqueness) of the "vertical" asymptote is visually apparent.

Halsey [1] has no formulas for the Lucas numbers corresponding to those for the Fibonacci numbers, i.e., (3.1) and (3.4). This is because the Pascal triangle generates the Fibonacci numbers but not the Lucas numbers. However, as is well known,

$$L_n = F_{n+1} + F_{n-1} \tag{4.1}$$

for integers. This carries over to real number subscripts, e.g., from Tables 1 and 3,

$$\begin{aligned} F_{7.8} + F_{9.8} &= 69.026763\ldots \quad \text{(to 6 decimal places)} \\ &= L_{8.8}. \end{aligned}$$

On this basis, one could combine $F_{\theta+1}$ and $F_{\theta-1}$ from (3.4) to obtain an integral expression for L_θ.

5. THE H CURVES

Putting $a = b = 1$ (i.e., $A = 2\alpha$, $B = 2\beta$) in (1.5), we have, from (1.6),

$$\begin{aligned} H_n &= F_n + L_n \qquad (5.1) \\ &= F_{n+1} - F_{n-1} + F_{n+1} + F_{n-1} \quad \text{by definition of } F_n \text{ and (4.1)} \\ &= 2F_{n+1}. \end{aligned}$$

Hence, a composite curve for $F_\theta + L_\theta$ is equivalent to the Fibonacci curve for $2F_{\theta+1}$. This H-curve ($a = 1$, $b = 1$) is drawn in Figure 3, where it is to be compared with the Fibonacci and Lucas curves in Figures 1 and 2, respectively.

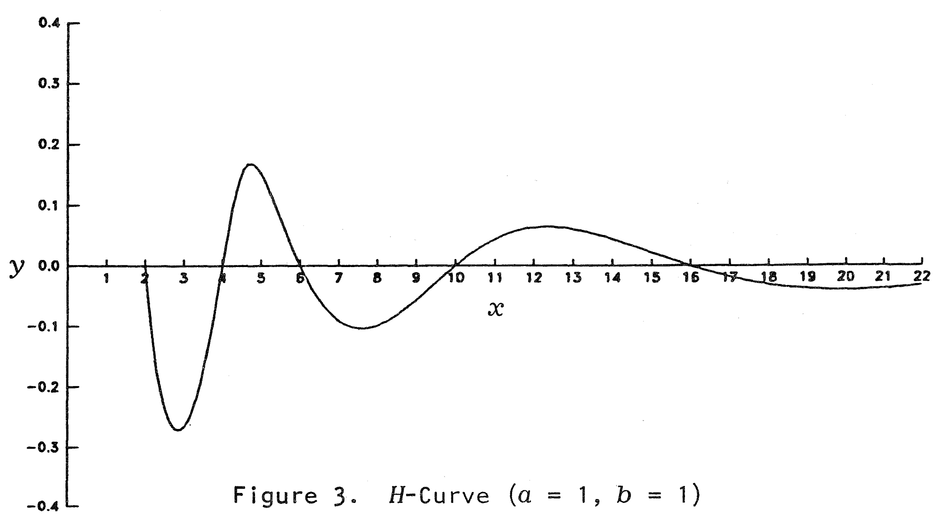

Figure 3. H-Curve ($a = 1$, $b = 1$)

Figure 3 might be taken as an illustration of the conclusion by Stein [5] regarding the intersection of Fibonacci sequences, e.g.,

$$\{F_n\} \cap \{L_n\} = 1,\ 3$$
$$\{F_n\} \cap \{F_n + L_n\} = 2$$
$$\{L_n\} \cap \{F_n + L_n\} = 4$$

Further, from (1.6),

$$\begin{aligned} H_n &= aF_n + bF_{n-1} + bF_{n+1} && \text{by (4.1)} \\ &= aF_n + bF_{n-1} + bF_n + bF_{n-1} && \text{by definition of } F_n \\ &= (a+b)F_n + 2bF_{n-1} \\ &= pF_n + qF_{n-1} \end{aligned} \tag{5.2}$$

where

$$p = a + b, \qquad q = 2b$$
$$= H_1 \qquad\quad = H_0 \quad \text{as in (1.1).}$$

ACKNOWLEDGMENT

Acknowledgment is gratefully made to Dr. E. W. Bowen, University of New England, Australia, for assistance with the computer-drawn graphs in Figures 1, 2, and 3. (These graphs are drawn from the equations of the curves, and supersede graphs which were drawn from the tables.)

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