AN EXPANSION OF x^m AND ITS COEFFICIENTS

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1. INTRODUCTION

This paper is concerned with an interesting expansion of x^m, where x and m are positive integers, and with the properties of its coefficients. One of the authors, Y. Imai, obtained expressions for 3^6 and 10^7 experimentally.

3^6 is systematically expressed by the sum of products below.

$$\begin{aligned}3^6 = {} & \frac{1}{6!}\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8 + \frac{57}{6!}\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7 \\ & + \frac{302}{6!}\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 + \frac{302}{6!}\cdot 0\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5 \\ & + \frac{57}{6!}\cdot (-1)\cdot 0\cdot 1\cdot 2\cdot 3\cdot 4 + \frac{1}{6!}\cdot (-2)\cdot (-1)\cdot 0\cdot 1\cdot 2\cdot 3.\end{aligned}$$

10^7 is systematically expressed by the sum of products below.

$$\begin{aligned}10^7 = {} & \frac{1}{7!}\cdot 10\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15\cdot 16 + \frac{120}{7!}\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15 \\ & + \frac{1191}{7!}\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13\cdot 14 + \frac{2416}{7!}\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13 \\ & + \frac{1191}{7!}\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12 + \frac{120}{7!}\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11 \\ & + \frac{1}{7!}\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10.\end{aligned}$$

To generalize the above expressions, we introduce a notion called the Z-coefficient. We note that Z is a number-theoretic function. We also note the following. If m and x are positive integers, then x^m can be expanded as follows:

$$x^m = \sum_{r=1}^{m}\left(\frac{Z(m,\ r)}{m!}\cdot\prod_{i=1}^{m}(x + i - r)\right).$$

The numerator $Z(m, r)$ is a number-theoretic function (we call it the Z-coefficient) defined by

$$Z(m,\ r) = \sum_{k=1}^{r}(-1)^{r+k}\cdot\binom{m+1}{r-k}\cdot k^m, \quad (r = 1,\ \ldots,\ m).$$

Another construction method for Z-coefficients $Z(m, r)$, $r = 1, \ldots, m$, and their properties will be given. The Z-coefficients have properties similar to those of the Pascal triangle.

2. PROPERTIES OF EXPANSIONS

These expansions have the following four properties:

1. In each case, the sum of these coefficients is equal to 1. That is:

$$\frac{1}{6!}+\frac{57}{6!}+\frac{302}{6!}+\frac{302}{6!}+\frac{57}{6!}+\frac{1}{6!} = 1,$$

$$\frac{1}{7!}+\frac{120}{7!}+\frac{1191}{7!}+\frac{2416}{7!}+\frac{1191}{7!}+\frac{120}{7!}+\frac{1}{7!} = 1.$$

If we denote these coefficients by $I(6, r)$ and $I(7, r)$, then

$$\sum_{r=1}^{6} I(6,\ r) = 1 \quad\text{and}\quad \sum_{r=1}^{7} I(7,\ r) = 1.$$

2. The denominators of these coefficients are 6! and 7! in these cases, respectively. Denoting the numerators of these coefficients by $Z(6, r)$, $r = 1, \ldots, 6$, and $Z(7, r)$, $r = 1, \ldots, 7$, we have

$$I(6,\ r) = \frac{Z(6,\ r)}{6!}\quad (r = 1,\ \ldots,\ 6),\qquad \sum_{r=1}^{6} Z(6,\ r) = 6!.$$

$$I(7,\ r) = \frac{Z(7,\ r)}{7!}\quad (r = 1,\ \ldots,\ 7),\qquad \sum_{r=1}^{7} Z(7,\ r) = 7!.$$

$Z(6, r)$ and $Z(7, r)$ are called Z-coefficients.

3. In both cases, Z-coefficients systematically distribute, i.e.,

1, 57, 302, 302, 57, 1 and 1, 120, 1191, 2416, 2291, 120, 1.

4. In the expressions for 3^6 and 10^7, the first members of each product except their coefficients are, respectively,

3, 2, 1, 0, -1, -2 and 10, 9, 8, 7, 6, 5, 4.

As is easily seen, the first integers of these descending sequences are 3 (the base of 3^6) and 10 (the base of 10^7).

The question now arises: Can we generalize the above properties?

3. THE COEFFICIENTS $Z(m, r)$ AND THE THEOREM

The answer to the question above is affirmative. We now have the following definition and theorem.

Definition: Let m and r be integers. $Z(m, r)$ is defined by

$$Z(m, r) = \sum_{k=1}^{r} (-1)^{r+k} \cdot \binom{m+1}{r-k} \cdot k^m, \quad (m \geqslant 1,\ r = 1,\ \ldots,\ m), \tag{1}$$

$$Z(m, r) = 0 \text{ for } m \leqslant 0 \text{ or } r \leqslant 0 \text{ or } m < r.$$

Theorem: Let x and m be positive integers. Then

$$\begin{aligned} x^m &= \frac{Z(m, 1)}{m!} \cdot x \cdot (x+1) \cdot (x+2) \cdot \cdots \cdot (x + (m-1)) \\ &\quad + \frac{Z(m, 2)}{m!} \cdot (x-1) \cdot x \cdot \cdots \cdot (x + (m-2)) \\ &\quad + \cdots + \frac{Z(m, m)}{m!} \cdot (x - (m-1)) \cdot \cdots \cdot x \\ &= \sum_{r=1}^{m} \left(\frac{Z(m, r)}{m!} \cdot \prod_{i=1}^{m} (x + i - r) \right). \end{aligned} \tag{2}$$

In order to prove the Theorem, we need the following Lemmas concerning the Z-coefficients.

Lemma 1: Let $Z(m, r)$ be Z-coefficients. Then:

$$Z(m+1, r) = (m - r + 2) \cdot Z(m, r-1) + r \cdot Z(m, r); \tag{3}$$

$$Z(m, r) = (m - r + 1) \cdot Z(m-1, r-1) + r \cdot Z(m-1, r); \tag{4}$$

$$Z(m+1, r+1) = (m - r + 1) \cdot Z(m, r) + (r+1) \cdot Z(m, r+1). \tag{5}$$

Proof of Lemma 1: It is clear that (3), (4), and (5) are equivalent to each other. We prove (5). By the definition of $Z(m, r)$, the right-hand side of (5) is written in the form

$$\sum_{k=1}^{r} \left((-1)^{r+k} \cdot (m - r + 1) \cdot \binom{m+1}{r-k} \cdot k^m \right) + \sum_{k=1}^{r} \left((-1)^{r+1+k} \cdot (r+1) \cdot \binom{m+1}{r+1-k} \cdot k^m \right) + (-1)^{2r+2} \cdot (r+1) \cdot \binom{m+1}{0} \cdot (r+1)^m.$$

A general term is expressed by the following:

$$(-1)^{r+k} \cdot (m - r + 1) \cdot \binom{m+1}{r-k} \cdot k^m + (-1)^{r+1+k} \cdot (r + 1) \cdot \binom{m+1}{r+1-k} \cdot k^m$$

$$= (-1)^{r+k} \cdot k^m \cdot \frac{(m+1)!}{(m + 1 - r + k)! \cdot (r + 1 - k)!} \cdot (-k) \cdot (m + 2)$$

$$= (-1)^{r+k+1} \cdot k^{m+1} \cdot \binom{m+2}{r-k+1}.$$

Therefore, the right-hand side of (5) is equal to

$$\sum_{k=1}^{r}\left((-1)^{r+k+1} \cdot \binom{m+2}{r-k+1} \cdot k^{m+1}\right) + (-1)^{2r+2} \cdot (r + 1) \cdot \binom{m+1}{0} \cdot (r + 1)^m,$$

which is

$$\sum_{k=1}^{r+1}\left((-1)^{r+k+1} \cdot \binom{m+2}{r-k+1} \cdot k^{m+1}\right).$$

By the definition of $Z(m, r)$, the last expression is equal to $Z(m + 1, r + 1)$. Hence, the proof is complete.

Lemma 2: Let $Z(m, r)$ be Z-coefficients. Then:

$$\sum_{r=1}^{m} Z(m, r) = m!, \quad (m \geqslant 1,\ r = 1, \ldots, m); \tag{6}$$

$$Z(m, r) = Z(m, m + 1 - r). \tag{7}$$

Equation (7) shows that Z-coefficients distribute symmetrically.

Proof of (6): By (4), the following equalities hold:

$$Z(m, 1) = m \cdot Z(m - 1, 0) + 1 \cdot Z(m - 1, 1),$$

$$Z(m, 2) = (m - 1) \cdot Z(m - 1, 1) + 2 \cdot Z(m - 1, 2),$$

$$Z(m, 3) = (m - 2) \cdot Z(m - 1, 2) + 3 \cdot Z(m - 1, 3),$$

$$\vdots \qquad \vdots$$

$$Z(m, m) = 1 \cdot Z(m - 1, m - 1) + m \cdot Z(m - 1, m).$$

From these equalities with $Z(m - 1, 0) = 0$ and $Z(m - 1, m) = 0$, we have

$$\sum_{r=1}^{m} Z(m, r) = m \cdot (Z(m - 1, 1) + \cdots + Z(m - 1, m - 1))$$

$$= m \cdot \sum_{r=1}^{m-1} Z(m - 1, r).$$

Hence, by the definition of $Z(1, 1)$,

$$\sum_{r=1}^{m} Z(m, r) = m \cdot (m - 1) \cdot \cdots \cdot 2 \cdot Z(1, 1) = m!.$$

Proof of (7): We prove (7) by induction on m. It is clear that (7) holds for $m = 1$. We assume that (7) holds for the positive integers not greater than m. We now show that (7) holds for $m + 1$, i.e.,

$$Z(m+1, r) = Z(m+1, m+2-r). \tag{8}$$

By (3), we have

$$Z(m+1, m+2-r) = r \cdot Z(m, m-r+1) + (m+2-r) \cdot Z(m, m+2-r).$$

By the induction hypothesis,

$$Z(m, m-r+1) = Z(m, r), \qquad Z(m, m+2-r) = Z(m, r-1).$$

Hence, by (3),

$$\begin{aligned} Z(m+1, m+2-r) &= r \cdot Z(m, r) + (m-r+2) \cdot Z(m, r-1) \\ &= Z(m+1, r). \end{aligned}$$

Therefore, (8) holds, as required.

Now, we return to the proof of the Theorem.

Proof of Theorem: We shall prove the Theorem by induction on m. It is clear that the Theorem holds for $m = 1$. We assume that (2) holds for positive integers not greater than m. We shall prove that (2) holds for $m + 1$, i.e.,

$$x^{m+1} = \frac{1}{(m+1)!} \cdot \sum_{r=1}^{m+1} \left(Z(m+1, r) \cdot \prod_{i=1}^{m+1} (x+i-r) \right). \tag{9}$$

By (3), we have

$$\begin{aligned} &\sum_{r=1}^{m+1} \left(Z(m+1, r) \cdot \prod_{i=1}^{m+1} (x+i-r) \right) \\ &\quad = \sum_{r=1}^{m+1} \left((m-r+2) \cdot Z(m, r-1) \cdot \prod_{i=1}^{m+1} (x+i-r) \right) \\ &\qquad\qquad + \sum_{r=1}^{m+1} \left(r \cdot Z(m, r) \cdot \prod_{i=1}^{m+1} (x+i-r) \right). \end{aligned}$$

Since $Z(m, r-1) = 0$ for $r = 1$ and $Z(m, r) = 0$ for $r = m + 1$, the right-hand side of the above is equal to

$$\begin{aligned} &\sum_{r=2}^{m+1} \left((m-r+2) \cdot Z(m, r-1) \cdot \prod_{i=1}^{m+1} (x+i-r) \right) \\ &\qquad\qquad + \sum_{r=1}^{m} \left((r \cdot Z(m, r) \cdot \prod_{i=1}^{m+1} (x+i-r) \right). \end{aligned}$$

Changing $r - 1$ to r in the first term, we have

$$\sum_{r=1}^{m}\left((m+1-r)\cdot Z(m,\ r)\cdot\prod_{i=1}^{m+1}(x+i-r-1)\right)$$

$$+\sum_{r=1}^{m}\left(r\cdot Z(m,\ r)\cdot\prod_{i=1}^{m+1}(x+i-r)\right)$$

$$=(m+1)\cdot\sum_{r=1}^{m}\left(Z(m,\ r)\cdot\prod_{i=1}^{m+1}(x+i-r-1)\right)$$

$$+\sum_{r=1}^{m}\left(r\cdot Z(m,\ r)\cdot\left(\prod_{i=1}^{m+1}(x+i-r)-\prod_{i=1}^{m+1}(x-1+i-r)\right)\right)$$

$$=(m+1)\cdot\sum_{r=1}^{m}\left(Z(m,\ r)\cdot\prod_{i=1}^{m+1}(x+i-r-1)\right)$$

$$+(m+1)\cdot\sum_{r=1}^{m}\left(r\cdot Z(m,\ r)\cdot\prod_{i=1}^{m}(x+i-r)\right)$$

$$=(m+1)\cdot x\cdot\sum_{r=1}^{m}\left(Z(m,\ r)\cdot\prod_{i=1}^{m}(x+i-r)\right).$$

By the induction hypothesis, the last expression is equal to

$$(m+1)!\cdot x^{m+1}.$$

Hence, (9) holds, as required.

4. REMARKS

4.1 If x and m $(x < m)$ are positive integers, then (2) is reduced as follows:

$$x^m = \frac{Z(m,\ 1)}{m!}\cdot x\cdot(x+1)\cdot\cdots\cdot(x+(m-1)) + \frac{Z(m,\ 2)}{m!}\cdot(x-1)\cdot x\cdot\cdots$$

$$\cdot(x+(m-2)) + \cdots + \frac{Z(m,\ x)}{m!}\cdot 1\cdot 2\cdot\cdots\cdot m$$

$$=\sum_{r=1}^{x}\left(\frac{Z(m,\ r)}{m!}\cdot\prod_{i=1}^{m}(x+i-r)\right).$$

4.2 Calculating $Z(m,\ r)$ for $1 \leqslant m \leqslant 6$, $r = 1, \ldots, m$, the following triangle is obtained:

$m = 1$	1					
$m = 2$	1	1				
$m = 3$	1	4	1			
$m = 4$	1	11	11	1		
$m = 5$	1	26	66	26	1	
$m = 6$	1	57	302	302	57	1

Clearly, this triangle is obtained by simple calculation. For example, to get

$Z(6, 3) = 302$, write all the values of $Z(5, r)$ $(r = 1, 2, 3, 4, 5)$ in one line from left to right (see the line for $m = 5$). Next, write r as a left subscript for $Z(5, r)$, i.e., ${}_rZ(5, r)$. Finally, write $5 - (r - 1)$ as the right subscript for $Z(5, r)$, i.e., $Z(5, r)_{5-(r-1)}$. Then, we obtain

$${}_1 1_5 \qquad {}_2 26_4 \qquad {}_3 66_3 \qquad {}_4 26_2 \qquad {}_5 1_1$$

$$302$$

$Z(6, 3) = 302 = 26 \cdot 4 + 3 \cdot 66$, which gives equation (4):

$$Z(m, r) = (m - r + 1) \cdot Z(m - 1, r - 1) + r \cdot Z(m - 1, r).$$

The symmetry of Z-coefficients is clear from the viewpoint of this construction method. The Pascal triangle is a special case of our triangle, i.e., the Pascal triangle is obtained by using 1 for all right- and left-hand subscripts. Let us call our triangle the "I-triangle."

4.3 By (6), it is clear that

$$\sum_{r=1}^{m} I(m, r) = \sum_{r=1}^{m} \frac{Z(m, r)}{m!} = 1.$$

4.4 It is an interesting problem to find the relation between Z-coefficients and Stirling numbers of the second kind (see [1]).

REFERENCE

1. M. Aigner. *Combinatorial Theory*. New York: Springer-Verlag, 1979.

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