GENERALIZED TRANSPOSABLE INTEGERS

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1. INTRODUCTION

Let x be an n-digit number expressed in base g; thus,

$$x = \sum_{i=0}^{n-1} a_i g^i \text{ with } 0 \leqslant a_i < g \text{ and } a_{n-1} \neq 0.$$

Let k be a positive integer. Then x is called k-transposable if and only if

$$kx = \sum_{i=0}^{n-2} a_i g^{i+1} + a_{n-1}. \tag{1}$$

Clearly, x is 1-transposable if and only if all of its digits are equal. Thus, we assume $k > 1$.

Kahan [2] studied decadic k-transposable integers. He showed that k must equal 3, that $x_1 = 142857$ and $x_2 = 285714$ are 3-transposable, and that all other 3-transposable integers are obtained by concatenating x_1 or x_2 m times, $m \geqslant 1$.

In [1], this author studied k-transposable integers for an arbitrary base g. Necessary and sufficient conditions were given for an n-digit, g-adic number to be k-transposable.

When a k-transposable integer is multiplied by k, its digits are shifted one place to the left with the leading digit moving to the units place. In this paper, we will generalize this shift of one place to a shift of j places, $1 \leqslant j < n$.

2. TRANSPOSABLE INTEGERS WITH ARBITRARY SHIFTS

We say that the n-digit number $x = \sum_{i=0}^{n-1} a_i g^i$ is a k-transposable, j-shift integer, or a (k, j)-integer for short, if and only if

$$kx = \sum_{i=0}^{n-1-j} a_i g^{i+j} + \sum_{i=n-j}^{n-1} a_i g^{i-(n-j)}, \text{ for } 1 \leqslant j < n \text{ and } 1 < k < g. \tag{2}$$

For example, again consider the decadic integers 142857 and 285714. Since

$6(142857) = 857142,$

$2(285714) = 571428,$

142857 is a (6, 3)-integer, while 285714 is a (2, 2)-integer.

We shall study (k, j)-integers for an arbitrary base g. Kahan [3] has determined all decadic n-digit $(k, n - 1)$-integers. He called these k-reverse transposable integers.

Rearranging the terms in (2), we get

$$(kg^{n-j} - 1)\sum_{i=n-j}^{n-1} a_i g^{i-(n-j)} = (g^j - k)\sum_{i=0}^{n-1-j} a_i g^i. \tag{3}$$

Let d be the greatest common divisor of $kg^{n-j} - 1$ and $g^j - k$. Then the following lemma is immediate.

Lemma 1: Let x be an n-digit, (k, j)-integer and let $d = (kg^{n-j} - 1, g^j - k)$. Then d satisfies the following:

(i) $(g, d) = 1$

(ii) $(k, d) = 1$

(iii) $k < d$

(iv) $g^n \equiv 1 \pmod{d}$

The following theorem gives necessary and sufficient conditions for the existence of (k, j)-integers.

Theorem 1: There exists an n-digit, (k, j)-integer if and only if there is an integer d with the following properties:

(i) $(k, d) = 1$

(ii) $k < d$

(iii) $d \mid g^j - k$

(iv) $g^n \equiv 1 \pmod{d}$

Proof: Lemma 1 shows that (i)-(iv) are necessary with $d = (kg^{n-j} - 1, g^j - k)$.

Now, suppose there exists a d satisfying (i)-(iv). Note that d divides $kg^{n-j} - 1$ since

$$kg^{n-j} - 1 \equiv g^j g^{n-j} - 1 \equiv g^n - 1 \equiv 0 \pmod{d}.$$

We now construct $\left[\frac{d}{k}\right]$ (k, j)-integers x_t. Let

$$x_t = \sum_{i=0}^{n-1} b_{t,i} g^i, \text{ with } t = 1, \ldots, \left[\frac{d}{k}\right].$$

The coefficients $b_{t,n-1}, \ldots, b_{t,n-j}$ are given by

$$\sum_{i=n-j}^{n-1} b_{t,i}\, g^{i-(n-j)} = \frac{g^j - k}{d}\, t. \tag{4}$$

We obtain (4) by dividing (3) by $g^j - k$ and requiring that $\sum_{n-j}^{n-1} b_{t,i}\, g^{i-(n-j)}$ be a multiple of $\frac{g^j - k}{d}$, since d divides $kg^{n-j} - 1$. Note that the highest power of g which occurs on each side of (4) is $j - 1$, so the coefficients $b_{t,i}$ are well defined. Using (3) we find that $b_{t,0}, \ldots, b_{t,n-j-1}$ are to be defined by

$$\sum_{i=0}^{n-1-j} b_{t,i}\, g^i = \frac{kg^{n-j} - 1}{d}\, t. \tag{5}$$

Equation (5) is also well defined, since $kt \leqslant d$.

We note here that the proof of Theorem 1 is a constructive one. The digits of k-transposable integers are found using (4) and (5). We now show that all g have (k, j)-integers.

Theorem 2: If $g = 5$ or $g \geqslant 7$, then g has a (k, j)-integer for all $j \geqslant 1$. If $g = 3$, 4, or 6, then g has a (k, j)-integer for $j \geqslant 2$.

Proof: If $g = 5$ or $g \geqslant 7$, choose k satisfying the following:

$$2 \leqslant k \leqslant g/2 \quad \text{and} \quad (k, g) = 1.$$

Then $d = g^j - k$, $j \geqslant 1$, satisfies (i)-(iii) of Theorem 1; further, $(d, g) = 1$. Hence, there exists n such that $g^n \equiv 1 \pmod{d}$. By Theorem 1, g has a (k, j)-integer.

For $g = 3$, 4, or 6, choose k such that

$$2 \leqslant k < g \quad \text{and} \quad (k, g) = 1.$$

Again, let $d = g^j - k$, $j \geqslant 2$, and apply Theorem 1. For these g, no $(k, 1)$-integers exist.

For j fixed, we now show that up to concatenation there are only a finite number of (k, j)-integers.

Theorem 3: Suppose $x = \sum_{i=0}^{n-1} a_i g^i$ is a (k, j)-integer. Let $d = (kg^{n-j} - 1, g^j - k)$ and let N be the order of g in U_d, the group of units of Z_d. Then x equals some (k, j)-integer concatenated n/N times.

Proof: Since $g^n \equiv 1 \pmod{d}$, n is a multiple of N. Let

$$x_t = \sum_{i=0}^{N-1} b_{t,i}\, g^i, \quad t = 1, \ldots, \left[\frac{d}{k}\right],$$

be the N-digit integers given by equations (4) and (5).

In (3), $\sum_{i=n-j}^{n-1} a_i g^{i-(n-j)}$ must be a multiple of $\frac{g^j - k}{d}$. Thus, for some t,

$$\sum_{i=n-j}^{n-1} a_i g^{i-(n-j)} = \left(\frac{g^j - k}{d}\right)t = \sum_{i=N-j}^{N-1} b_{t,i} g^{i-(N-j)}$$

so

$$a_{n-i} = b_{t,N-i}, \text{ for } i = 1, \ldots, j.$$

Thus,

$$\sum_{i=0}^{n-1-j} a_i g^i = \left(\frac{kg^{n-j} - 1}{d}\right)t = g^{n-N}\left(\frac{kg^{N-j} - 1}{d}\right)t + \left(\frac{g^{n-N} - 1}{d}\right)t.$$

Note that $kt \leqslant d$. Now, since

$$\sum_{i=0}^{N-1-j} b_{t,i} g^i = \left(\frac{kg^{N-j} - 1}{d}\right)t,$$

we must have

$$a_{n-i} = b_{t,N-i}, \quad i = j + 1, \ldots, N.$$

Further,

$$\left(\frac{g^{n-N} - 1}{d}\right)t = \left(\frac{g^j - k}{d}\right)tg^{n-N-j} + \left(\frac{kg^{n-N-j} - 1}{d}\right)t.$$

Hence,

$$\sum_{i=n-N-j}^{n-N-1} a_i g^i = \left(\frac{g^j - k}{d}\right)tg^{n-N-j}$$

or

$$\sum_{i=n-N-j}^{n-N-1} a_i g^{i-(n-N-j)} = \left(\frac{g^j - k}{d}\right)t = \sum_{i=N-j}^{N-1} b_{t,i} g^{i-(N-j)}.$$

Thus, $a_{n-N-i} = b_{t,N-i}$, $i = 1, \ldots, j$, and $a_{n-N-i} = b_{t,N-i}$, $i = j - 1, \ldots, N$. Continuing, we find that x equals x_t concatenated n/N times.

3. $(k, 1)$-INTEGERS ARE ALSO (ℓ, j)-INTEGERS

In some cases $(k, 1)$-integers are also (ℓ, j)-integers. Consider the multiples of the decadic $(3, 1)$-integer $y = 142857$:

$$2y = 285714; \quad 4y = 571428; \quad 5y = 714285; \quad 6y = 857142.$$

Thus, y is also a (2, 2), (4, 4), (5, 5), and (6, 3)-integer. We observe that y is an (ℓ, j)-integer when $\ell \equiv 3^j \pmod 7$. Here $7 = d = (g - k, kg^{n-1} - 1)$, with $g = 10$, $k = 3$, and $n = 6$. We will show that this is always the case when ℓy is an n-digit number. The following lemmas will be useful.

Lemma 2: Suppose $x = \sum_{i=0}^{n-1} a_i g^i$ is a $(k, 1)$-integer. Let $d = (g - k, kg^{n-1} - 1)$.

Then

$$dx = \frac{d}{g - k} a_{n-1}(g^n - 1).$$

Proof: Since d divides $g - k$, $d = \frac{g - k}{r}$ for some r. Thus, we have:

$$d\sum_{i=0}^{n-1} a_i g^i = \frac{1}{r}(g - k)\sum_{i=0}^{n-1} a_i g^i = \frac{1}{r}\left[\sum_{i=0}^{n-1} a_i g^{i+1} - \sum_{i=0}^{n-2} a_i g^{i+1} - a_{n-1}\right]$$

$$= \frac{1}{r} a_{n-1}(g^n - 1) = \frac{d}{g - k} a_{n-1}(g^n - 1).$$

Lemma 3: Suppose $x = \sum_{i=0}^{n-1} a_i g^i$ is a $(k, 1)$-integer. Then, for $j \geqslant 2$, we have

$$k^j x = \sum_{i=0}^{n-j-1} a_i g^{i+j} + \sum_{i=n-j}^{n-1} a_i g^{i-(n-j)} + r_j(g^n - 1),$$

where

$$r_j = \sum_{i=2}^{j} (a_{n-i} - k^{i-1} a_{n-1}) g^{j-i}.$$

Proof: The proof is by induction. Since the initial step with $j = 2$ is similar to the induction step, we will do only the latter. Consider

$$k^{j+1}x = k^j\left(\sum_{i=0}^{n-2} a_i g^{i+1} + a_{n-1}\right) = gk^j\sum_{i=0}^{n-1} a_i g^i - k^j a_{n-1}(g^n - 1)$$

$$= g\left[\sum_{i=0}^{n-j-1} a_i g^{i+j} + \sum_{i=n-j}^{n-1} a_i g^{i-(n-j)} + r_j(g^n - 1)\right] - k^j a_{n-1}(g^n - 1)$$

$$= \sum_{i=0}^{n-j-2} a_i g^{i+j+1} + \sum_{i=n-j-1}^{n-1} a_i g^{i-(n-j-1)}$$

$$+ (a_{n-j-1} - k^j a_{n-1})(g^n - 1) + r_j g(g^n - 1)$$

$$= \sum_{i=0}^{n-j-2} a_i g^{i+j+1} + \sum_{i=n-j-1}^{n-1} a_i g^{i-(n-j-1)} + r_{j+1}(g^n - 1).$$

Theorem 4: Suppose that $x = \sum_{i=0}^{n-1} a_i g^i$ is a $(k, 1)$-integer. Let $d = (g - k, kg^{n-1} - 1)$. Suppose ℓx is an n-digit number with $\ell < d$. Then x is an (ℓ, j)-integer if $\ell \equiv k^j \pmod{d}$.

Proof: Since $\ell \equiv k^j \pmod{d}$, $\ell = k^j - sd$ for some nonnegative integer s. Then by Lemmas 2 and 3,

$$\ell x = \sum_{i=0}^{n-j-1} a_i g^{i+j} + \sum_{i=n-j}^{n-1} a_i g^{i-(n-j)} + \left(r_j - s\frac{d}{g - k} a_{n-1}\right)(g^n - 1).$$

Since ℓx is an n-digit number, $r_j - s\frac{d}{g - k} a_{n-1}$ must equal zero. Hence, x is an (ℓ, j)-integer.

While $(k, 1)$-integers give rise to (ℓ, j)-integers, an (ℓ, j)-integer need not be a $(k, 1)$-integer. For example, the decadic number 153846 is a (4, 5)-integer, but it is not a $(k, 1)$-integer for any k.

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