GENERALIZED GAUSSIAN LUCAS PRIMORDIAL FUNCTIONS

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1. INTRODUCTION

The Fibonacci numbers F_n are defined as $F_0 = 0$, $F_1 = 1$ with the successive numbers given by the recurrence relation $F_{n+2} = F_{n+1} + F_n$.

Horadam[6] extended these numbers to the complex number field by defining them as $F_n^* = F_n + iF_{n+1}$.

Taking a different approach, Berzsenyi [2] defined the set of complex numbers at the Gaussian integers and called them the Gaussian Fibonacci Numbers. He defined them as follows: Let $n \in \mathbb{Z}$ and m be a nonnegative integer. Then, the Gaussian Fibonacci numbers $F(n, m)$ are defined as

$$F(n, m) = \sum_{k=0}^{m} \binom{m}{k} i^k F_{n-k},$$

where F_j are the (real) Fibonacci numbers defined above. He proved that

$$F(n, m) = F(n - 1, m) + F(n - 2, m), \quad n \geqslant 2.$$

This relation implies that any adjacent triplets on the horizontal line possess a Fibonacci-type recurrence relation. In a paper in 1981, Harman (see [4]) elaborated Berzsenyi's idea and defined another set of complex numbers by directly using the Fibonacci recurrence relation. He defined them as follows:

Let $(n, m) = n + im$, where $n, m \in \mathbb{Z}$. The complex Fibonacci numbers denoted by $G(n, m)$ are those which satisfy

$$G(0, 0) = 0,\ G(0, 1) = 1,\ G(1, 0) = i,\ G(1, 1) = 1 + i,$$

and

$$G(n + 2, m) = G(n + 1, m) + G(n, m),$$
$$G(n, m + 2) = G(n, m + 1) + G(n, m).$$

The initial values and the recurrence relations are sufficient to specify uniquely the value of $G(n, m)$ for each (n, m) in the plane. It is easy to see that

$$G(n, 0) = F_n \quad \text{and} \quad G(0, m) = iF_m.$$

The advantage of Harman's definition over Berzsenyi's is threefold:

1. While in Berzsenyi's definition, any adjacent horizontal triplets in the plane satisfy the Fibonacci recurrence relation, in Harman's definition, any adjacent horizontal *and* vertical triplets do the same.
2. Horadam's complex Fibonacci numbers F_n^* come as a special case for Harman's. Indeed, $F_m^* = G(1, m)$.
3. By obtaining a recurrence relation for $G(n, m)$ itself, Harman was able to prove some new summation identities for $\{F_n\}$.

Pethe, in collaboration with Horadam, extended Harman's idea to define Generalized Gaussian Fibonacci Numbers [10]. They again denoted these numbers by $G(n, m)$ and defined them at the Gaussian integers (n, m) as follows: Let p_1, p_2 be two fixed nonzero real numbers. Define

$$G(0, 0) = 0,\ G(1, 0) = 1,\ G(0, 1) = i,\ G(1, 1) = p_2 + ip_1,$$

with the conditions $G(n + 2, m) = p_1G(n + 1, m) - q_1G(n, m)$, and $G(n, m + 2) = p_2G(n, m + 1) - q_2G(n, m)$.

With the help of this extension of Harman's definition, the authors were able to obtain a wealth of summation identities involving the combinations of Fibonacci numbers and polynomials, Pell numbers and polynomials, and Chebyshev polynomials of the second kind. Observe that these numbers and polynomials all have the first two initial values as 0 and 1. Consequently, it is natural to ask, as in Remark 4 of [10], if a further extension that would include numbers and polynomials whose first two initial values were other than 0 and 1 is possible. The positive answer to this question is precisely the object of this paper.

Our main result is Theorem 6.1. With the help of a single equation, (6.1) of this theorem, various summation identities involving the product terms of Fermat's numbers, Fibonacci numbers and polynomials, Pell numbers and polynomials, Lucas numbers and polynomials, and Chebyshev polynomials of the first and second kinds are obtained. Besides these identities, (6.1) has the potential for obtaining many more by varying the values of m and n. The extension, first thought to be straightforward, did not turn out to be so. It still had to be formulated in terms of the Lucas fundamental sequence [9] whose first two terms are 0 and 1.

2. PRELIMINARIES

Let $\{U_n\}$ and $\{W_n\}$ denote the sequences defined as follows,

$$U_0 = 0,\ U_1 = 1,\ U_{n+2} = pU_{n+1} - qU_n, \quad n \geqslant 0,$$
$$W_0 = a,\ W_1 = b,\ W_{n+2} = pW_{n+1} - qW_n, \quad n \geqslant 0,$$

where a, b, p, and q are any real numbers, p, $q \neq 0$. The sequence $\{U_n\}$ is the fundamental sequence defined by Lucas and $\{W_n\}$ is the one defined and extensively studied by Horadam (see [9], [7], and [8]). Lucas's primordial function is the special case of $\{W_n\}$ with $W_0 = 2$ and $W_1 = p$. The relation between the terms of $\{W_n\}$ and $\{U_n\}$ is given by

$$W_n = bU_n - aqU_{n-1}. \tag{2.1}$$

Let $\{V_n\}$ be the complex-valued variant of Horadam's sequence defined by $V_0 = a$, $V_1 = ib$, with the recurrence relation $V_{n+2} = pV_{n+1} - qV_n$.

As above, it is clear that

$$V_n = ibU_n - aqU_{n-1}. \tag{2.2}$$

3. DEFINITION

Let (n, m), $n, m \in \mathbb{Z}$, denote the set of Gaussian integers $(n, m) = n + im$. Further, let

$$G: (n, m) \to \mathbb{C},$$

where $\mathbb{C}$ is the set of complex numbers, be the function defined as follows.

For fixed real numbers p and q, define

$$G(0, 0) = a,\ G(1, 0) = b,\ G(0, 1) = ib,\ G(1, 1) = pb(1 + i) \tag{3.1}$$

with the following conditions:

$$G(n + 2, m) = pG(n + 1, m) - qG(n, m), \tag{3.2}$$

and

$$G(n, m + 2) = pG(n, m + 1) - qG(n, m). \tag{3.3}$$

Conditions (3.2) and (3.3) with the initial values (3.1) are sufficient to obtain a unique value for every Gaussian integer.

4. EXPRESSION FOR $G(n, m)$

Lemma 4.1: We have

$$G(n, 0) = W_n, \ G(0, m) = V_n. \tag{4.1}$$

Proof: The proof is simple and, therefore, omitted here.

Remark: Observe that if $a = 0$ and $b = 1$, the definition for $G(n, m)$ reduces to that of Pethe & Horadam's "Generalised Gaussian Fibonacci Numbers" [10], where $p_1 = p_2 = p$ and $q_1 = q_2 = q$. Further, if $a = 0$, $b = 1$, and $p = 1$, $q = 1$, this definition reduces to Harman's "Complex Fibonacci Numbers" [4].

Theorem 4.2: $G(n, m)$ is given by

$$G(n, m) = bU_nU_{m+1} + aq^2U_{n-1}U_{m-1} + ibU_{n+1}U_m. \tag{4.2}$$

Proof: We use induction for the proof. Suppose (4.2) holds for all integers $0, 1, \ldots, n$ for the first number in the ordered pair (n, m) and for all integers $0, 1, \ldots, m$ for the second number. By (3.2), we have

$$G(n + 1, m) = pG(n, m) - qG(n - 1, m). \tag{4.3}$$

Applying (4.2) to the right side of (4.3), we obtain

$$\begin{aligned} G(n + 1, m) &= p[bU_nU_{m+1} + aq^2U_{n-1}U_{m-1} + ibU_{n+1}U_m] \\ &\qquad - q[bU_{n-1}U_{m+1} + aq^2U_{n-2}U_{m-1} + ibU_nU_m] \\ &= b(pU_n - qU_{n-1})U_{m+1} + aq^2(pU_{n-1} - qU_{n-2})U_{m-1} \\ &\qquad + ib(pU_{n+1} - qU_n)U_m. \end{aligned}$$

Therefore, by the recurrence relation of $\{U_n\}$, we get

$$G(n + 1, m) = bU_{n+1}U_{m+1} + aq^2U_nU_{m-1} + ibU_{n+2}U_m. \tag{4.4}$$

The right side of (4.4) is exactly the right side of (4.2) with n replaced by $n + 1$. Similarly, we prove that

$$G(n, m + 1) = bU_nU_{m+2} + aq^2U_{n-1}U_m + ibU_{n+1}U_{m+1}. \tag{4.5}$$

By (4.4), (4.5), and the induction principle, (4.2) holds for all nonnegative integers.

5. RECURRENCE RELATION FOR $G(n, m)$

Theorem 5.1: For fixed n and m, the recurrence relation for $G(n, m)$ is given by

$$G(n+2k+s,\ m+2k+s) = bp(1+i)\sum_{j=1}^{2k}(-1)^j(q)^{2k-j}U_{n+j+s}U_{m+j+s} \tag{5.1}$$

$$+\ apq^2\sum_{j=1}^{2k}(-1)^j(q)^{2k-j}U_{n+j-2+s}U_{m+j-1+s} + q^{2k}G(n+s,\ m+s),$$

where $s = 0$ or 1.

Proof: For the proof, we again use induction on k. First we find the expressions for $G(n+2,\ m+2)$ and $G(n+3,\ m+3)$. By (4.2), we have

$$\begin{aligned}
G(n+2,\ m+2) &= bU_{n+2}U_{m+3} + aq^2U_{n+1}U_{m+1} + ibU_{n+3}U_{m+2}\\
&= bU_{n+2}(pU_{m+2} - qU_{m+1}) + aq^2(pU_n - qU_{n-1})(pU_m - qU_{m-1})\\
&\quad + ib(pU_{n+2} - qU_{n+1})U_{m+2}\\
&= bp(1+i)U_{n+2}U_{m+2} - bqU_{n+2}U_{m+1} - ibqU_{n+1}U_{m+2}\\
&\quad + aq^2(p^2U_nU_m - pqU_nU_{m-1} - pqU_{n-1}U_m + q^2U_{n-1}U_{m-1})\\
&= bp(1+i)U_{n+2}U_{m+2} - bq(pU_{n+1} - qU_n)U_{m+1} - ibqU_{n+1}(pU_{m+1} - qU_m)\\
&\quad + aq^2(p^2U_nU_m - pqU_nU_{m-1} - pqU_{n-1}U_m + q^2U_{n-1}U_{m-1})\\
&= bp(1+i)(U_{n+2}U_{m+2} - qU_{n+1}U_{m+1}) + ap^2q^2U_nU_m - apq^3U_{n-1}U_m\\
&\quad - apq^3U_nU_{m-1} + q^2(bU_nU_{m+1} + aq^2U_{n-1}U_{m-1} + ibU_{n+1}U_m)\\
&= bp(1+i)(U_{n+2}U_{m+2} - qU_{n+1}U_{m+1}) + apq^2U_n(pU_m - qU_{m-1})\\
&\quad - apq^3U_{n-1}U_m + q^2G(n,\ m).
\end{aligned}$$

Using the recurrence relation for $\{U_m\}$ once again, we finally obtain

$$\begin{aligned}
G(n+2,\ m+2) &= bp(1+i)(U_{n+2}U_{m+2} - qU_{n+1}U_{m+1})\\
&\quad + apq^2(U_nU_{m+1} - qU_{n-1}U_m) + q^2G(n,\ m),
\end{aligned} \tag{5.2}$$

which is the same as (5.1) when $k = 1$ and $s = 0$.

Replacing n and m by $n + 1$ and $m + 1$, respectively, in (5.2) we have

$$\begin{aligned}
G(n+3,\ m+3) &= bp(1+i)(U_{n+3}U_{m+3} - qU_{n+2}U_{m+2})\\
&\quad + apq^2(U_{n+1}U_{m+2} - qU_nU_{m+1}) + q^2G(n+1,\ m+1).
\end{aligned} \tag{5.3}$$

Again, it is easily seen that (5.3) is exactly the same as (5.1) when $k = 1$ and $s = 1$. Thus, (5.1) holds for the initial values $k = 1$, $s = 0$, and $k = 1$, $s = 1$. Suppose next that (5.1) holds for, and up to, some positive integer k. We will show, then, that it also holds for $k + 1$. First let $s = 0$. Now, although n and m are assumed to be fixed in (5.2), it is clear that (5.2) is true for any positive integers n and m. Therefore, we can write the expression for $G(n + 2k + 2,\ m + 2k + 2)$ by replacing n and m in (5.2) by $n + 2k$ and $m + 2k$, respectively. Thus, we have

$$G(n+2k+2,\ m+2k+2) = bp(1+i)(U_{n+2k+2}U_{m+2k+2} - qU_{n+2k+1}U_{m+2k+1}) + apq^2(U_{n+2k}U_{m+2k+1} - qU_{n+2k-1}U_{m+2k}) + q^2G(n+2k,\ m+2k). \tag{5.4}$$

Using (5.1) for $s = 0$ in (5.4), we get

$$\begin{aligned}G(n+2k+2,\ m+2k+2) &= bp(1+i)(U_{n+2k+2}U_{m+2k+2} - qU_{n+2k+1}U_{m+2k+1})\\ &+ apq^2(U_{n+2k}U_{m+2k+1} - qU_{n+2k-1}U_{m+2k})\\ &+ q^2\Big\{bp(1+i)\sum_{j=1}^{2k}(-1)^j(q)^{2k-j}U_{n+j}U_{m+j}\\ &\quad + apq^2\sum_{j=1}^{2k}(-1)^j(q)^{2k-j}U_{n+j-2}U_{m+j-1} + q^{2k}G(n,\ m)\Big\}.\end{aligned}$$

Combining the first four terms on the right with the corresponding terms in the braces, we have

$$\begin{aligned}G(n+2k+2,\ m+2k+2) &= bp(1+i)\sum_{j=1}^{2k+2}(-1)^j(q)^{2k+2-j}U_{n+j}U_{m+j}\\ &+ apq^2\sum_{j=1}^{2k+2}(-1)^j(q)^{2k+2-j}U_{n+j-2}U_{m+j-1} + q^{2k+2}G(n,\ m).\end{aligned} \tag{5.5}$$

Identity (5.5) shows that (5.1) with $s = 0$ is true if k is replaced by $k + 1$. Similarly, we can show that (5.5) with $s = 1$ also holds if k is replaced by $k + 1$. Induction on k then shows that (5.1) holds for all k when k is a positive integer.

6. IDENTITY FOR THE SEQUENCE $\{W_n\}$

Equation (5.1) enables us to prove an important identity involving the product terms of the sequences $\{W_n\}$ and $\{U_n\}$. We prove

Theorem 6.1:

$$\sum_{j=1}^{N} p(-1)^{j+1}(q)^{N-j}W_{n+j}U_{m+j} = q^N W_n U_{m+1} + \begin{cases} W_{n+N+1}U_{m+N}, & N \text{ odd},\\ -W_{n+N}U_{m+N+1}, & N \text{ even}.\end{cases} \tag{6.1}$$

Proof: Equating the real and imaginary parts of (5.1), we get

$$\begin{aligned}&bp\sum_{j=1}^{2k}(-1)^j(q)^{2k-j}U_{n+j+s}U_{m+j+s} + apq^2\sum_{j=1}^{2k}(-1)^j(q)^{2k-j}U_{n+j-2+s}U_{m+j-1+s}\\ &= bU_{n+2k+s}U_{m+2k+1+s} + aq^2U_{n+2k-1+s}U_{m+2k-1+s}\\ &\quad - q^{2k}(bU_{n+s}U_{m+1+s} + aq^2U_{n-1+s}U_{m-1+s})\end{aligned} \tag{6.2}$$

and

$$bp\sum_{j=1}^{2k}(-1)^j(q)^{2k-j}U_{n+j+s}U_{m+j+s} = bU_{n+2k+1+s}U_{m+2k+s} - q^{2k}bU_{n+1+s}U_{m+s}. \tag{6.3}$$

Note that if $a = 0$ and $b = 1$, (6.2) and (6.3) reduce, respectively, to (5.1) and (5.2) of [10], where $p_2 = p_1 = p$ and $q_2 = q_1 = q$.

To convert identity (6.2) to the one containing the terms of the sequence $\{W_n\}$, we proceed as follows.

The left-hand side of (6.2) equals

$$p\sum_{j=1}^{2k-1}(-1)^j(q)^{2k-j}(bU_{n+j+s} - aqU_{n+j-1+s})U_{m+j+s} + bpU_{n+2k+s}U_{m+2k+s} - apq^{2k+1}U_{n-1+s}U_{m+s}. \tag{6.4}$$

Using (2.1) in (6.4), we see that the left-hand side of (6.2) equals

$$p\sum_{j=1}^{2k-1}(-1)^j(q)^{2k-j}W_{n+j+s}U_{m+j+s} + bpU_{n+2k+s}U_{m+2k+s} - apq^{2k+1}U_{n-1+s}U_{m+s}.$$

Therefore, equation (6.2), after rearranging terms, becomes

$$\begin{aligned}
&\sum_{j=1}^{2k-1}p(-1)^j(q)^{2k-j}W_{n+j+s}U_{m+j+s}\\
&= bU_{n+2k+s}U_{m+2k+1+s} - bpU_{n+2k+s}U_{m+2k+s} + aq^2U_{n+2k-1+s}U_{m+2k-1+s}\\
&\qquad + apq^{2k+1}U_{n-1+s}U_{m+s} - aq^{2k+2}U_{n-1+s}U_{m-1+s} - bq^{2k}U_{n+s}U_{m+1+s}\\
&= bU_{n+2k+s}(U_{m+2k+1+s} - pU_{m+2k+s}) + aq^2U_{n+2k-1+s}U_{m+2k-1+s}\\
&\qquad + aq^{2k+1}U_{n-1+s}(pU_{m+s} - qU_{m-1+s}) - bq^{2k}U_{n+s}U_{m+1+s}\\
&= bU_{n+2k+s}(-qU_{m+2k-1+s}) + aq^2U_{n+2k-1+s}U_{m+2k-1+s} + aq^{2k+1}U_{n-1+s}U_{m+1+s}\\
&\qquad - bq^{2k}U_{n+s}U_{m+1+s}\\
&= -q(bU_{n+2k+s} - aqU_{n+2k-1+s})U_{m+2k-1+s} - q^{2k}(bU_{n+s} - aqU_{n-1+s})U_{m+1+s}.
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{2k-1}(-1)^{j+1}pq^{2k-j-1}W_{n+j+s}U_{m+j+s} = W_{n+2k+s}U_{m+2k-1+s} + q^{2k-1}W_{n+s}U_{m+1+s}. \tag{6.5}$$

Putting $s = 0$ in (6.5), adding $-pW_{n+2k}U_{m+2k}$ to both sides of (6.5), and then using the recurrence relation for $\{U_n\}$, we get

$$\sum_{j=1}^{2k}(-1)^{j+1}pq^{2k-j}W_{n+j}U_{m+j} = -W_{n+2k}U_{m+2k+1} + q^{2k}W_nU_{m+1}. \tag{6.6}$$

Replacing $2k - 1$ and $2k$ in, respectively (6.5) with $s = 0$ and (6.6) by N, we finally obtain (6.1).

7. APPLICATION TO SOME SEQUENCES

7.1 Arithmetic Progression: Let $p = 2$ and $q = 1$. Taking $U_0 = 0$, $U_1 = 1$, and $W_0 = a$, $W_1 = a + d$, it is easily seen that $\{W_n\}$ becomes an arithmetic progression $\{A_n\}$ and $\{U_n\}$, the sequence of nonnegative integers, where $U_n = n$. Equation (6.1) reduces to

$$\sum_{j=1}^{N} 2(-1)^{j+1}(m+j)A_{n+j} = (m+1)A_n + \begin{cases} (m+N)A_{n+N+1}, & N \text{ odd}, \\ -(m+N+1)A_{n+N}, & N \text{ even}. \end{cases} \tag{7.1}$$

7.2 Geometric Progression: Let $p = q+1$, $W_0 = a$, and $W_1 = aq$. Consequently, the sequence $\{W_n\}$ becomes the geometric progression with common ratio q and $W_n = aq^n$, and the sequence $\{U_n\}$ with $U_0 = 0$ and $U_1 = 1$ has the n^{th} term U_n given by

$$U_n = \sum_{j=0}^{n-1} q^j, \quad n = 1, 2, \ldots .$$

Let us denote the geometric sequence $\{W_n\}$ by $\{G_n^{(q)}\}$. Equation (6.1) reduces to

$$(q+1)\sum_{j=1}^{N}(-1)^{j+1}q^{N-j}G_{n+j}^{(q)}U_{m+j} = q^N G_n^{(q)}U_{m+1} + \begin{cases} G_{n+N+1}^{(q)}U_{m+N}, & N \text{ odd}, \\ -G_{n+N}^{(q)}U_{m+N+1}, & N \text{ even}. \end{cases} \tag{7.2}$$

7.3 Fermat's Sequence: Let $p = 3$, $q = 2$, $W_0 = 2$, and $W_1 = 3$. Then $\{W_n\}$ is Fermat's sequence (see [7]). Let us denote it by $\{M_n\}$. With these values of p and q, $\{U_n\}$ is easily seen to be the sequence given by $U_n = 2^n - 1$. Equation (6.1) reduces to

$$3\sum_{j=1}^{N}(-1)^{j+1}2^{N-j}M_{n+j}U_{m+j} = 2^N M_n U_{m+1} + \begin{cases} M_{n+N+1}U_{m+N}, & N \text{ odd}, \\ -M_{n+N}U_{m+N+1}, & N \text{ even}. \end{cases} \tag{7.3}$$

Remark: In fact, $\{U_n\}$ is also known as Fermat's sequence. M_n and U_n are given by

$$M_n = 2^n + 1 \quad \text{and} \quad U_n = 2^n - 1.$$

7.4 Fibonacci and Pell Polynomials: Next, let $p = x$ and $q = -1$. Then, with $W_0 = 1$ and $W_1 = x$, $\{W_n\}$ reduces to the Fibonacci sequence $\{F_n(x)\}$, and with $U_0 = 0$ and $U_1 = 1$, $\{U_n\}$ becomes the Pell polynomial sequence $\{P_n(x)\}$, see [5]. It is easy to see that for $x = 1$ and $x = 2$, $\{U_n\}$ reduces to Fibonacci and Pell numbers, respectively, see [5]. Equation (6.1) becomes

$$\sum_{j=1}^{N} xF_{n+j}(x)P_{m+j}(x) = -F_n(x)P_{m+1}(x) + \begin{cases} F_{n+N+1}(x)P_{m+N}(x), & N \text{ odd}, \\ F_{n+N}(x)P_{m+N+1}(x), & \text{even}. \end{cases} \tag{7.4}$$

Remark on Lucas Polynomials: If $p = x$, $q = -1$, $W_0 = x$, and $W_1 = x^2 + 2$, $\{W_n\}$ reduces to the Lucas polynomial sequence $\{L_n(x)\}$ [5]. Since p, q and $\{U_n\}$ are the same as in section 7.4 above, equation (6.1) reduces to (7.4), where F_n is changed to L_n, that is

$$\sum_{j=1}^{N} xL_{n+j}(x)P_{m+j}(x) = -L_n(x)P_{m+1}(x) + \begin{cases} L_{n+N+1}(x)P_{m+N}(x), & N \text{ odd}, \\ L_{n+N}(x)P_{m+N+1}(x), & N \text{ even}. \end{cases} \tag{7.5}$$

7.5 Chebyshev Polynomials: Now let $p = 2x$, $q = 1$, $W_0 = 1$, and $W_1 = x$. Then $W_n(x)$ reduces to the n^{th} Chebyshev polynomial $T_n(x)$ of the first kind and $U_n(x)$ reduces to $S_n(x)$, that of the second kind [1], where

$$T_n(x) = \cos n\theta,\ S_n(x) = \frac{\sin n\theta}{\sin \theta}, \text{ and } \theta = \cos^{-1}x.$$

From (6.1), we obtain

$$\sum_{j=1}^{N} 2(-1)^{j+1} xT_{n+j}(x)S_{m+j}(x) = T_n(x)S_{m+1}(x) + \begin{cases} T_{n+N+1}(x)S_{m+N}(x), & N \text{ odd}, \\ -T_{n+N}(x)S_{m+N+1}(x), & N \text{ even}. \end{cases} \tag{7.6}$$

8. SPECIAL NUMERICAL CASES

Results of section 7 are more comprehensible and more interesting for some particular values of n and m. These are listed below. Some of these identities are known, and some appear to be new.

(A) $n = 0,\ m = 0$

$$\sum_{j=1}^{N} (-1)^{j+1} jA_j = \frac{a}{2} + \begin{cases} \frac{1}{2}NA_{N+1}, & N \text{ odd}, \\ -\frac{1}{2}(N+1)A_N, & N \text{ even}, \end{cases} \tag{7.1}^*$$

where $A_0 = a$ is the first term of the arithmetic progression $\{A_n\}$.

$$\sum_{j=1}^{N} (-1)^{j+1} q^{N-j}(q+1)G_j^{(q)}U_j = aq^N + \begin{cases} G_{N+1}^{(q)}U_N, & N \text{ odd}, \\ -G_N^{(q)}U_{N+1}, & N \text{ even}, \end{cases} \tag{7.2}^*$$

where a is the first term of the geometric progression $\{G_n^{(q)}\}$. Using the fact that $G_n^{(q)} = aq^n$, we find that (7.2)* reduces to

$$\sum_{j=1}^{N} (-1)^{j+1}(q+1)U_j = 1 + \begin{cases} qU_N, & N \text{ odd}, \\ -U_{N+1}, & N \text{ even}. \end{cases}$$

Observing that in (7.3) $U_n = 2^n - 1$, we see that (7.3), with $n = 0$, $m = 0$, reduces to

$$\sum_{j=1}^{N} (-1)^{j+1} 2^{N-j}(2^j - 1)M_j = \frac{1}{3}\left[2^{N+1} + \begin{cases} (2^N - 1)M_{N+1}, & N \text{ odd}, \\ -(2^{N+1} - 1)M\ , & N \text{ even}; \end{cases}\right] \tag{7.3}^*$$

$$\sum_{j=1}^{N} xF_j(x)P_j(x) = -1 + \begin{cases} F_{N+1}(x)P_N(x), & \text{odd}, \\ F_N(x)P_{N+1}(x), & \text{even}; \end{cases} \tag{7.4}^*$$

$$\sum_{j=1}^{N} xL_j(x)P_j(x) = - \quad + \begin{cases} L_{N+1}(x)P_N(x), & N \text{ odd}, \\ L_N(x)P_{N+1}(x), & N \text{ even}; \end{cases} \tag{7.5*}$$

$$\sum_{j=1}^{N} (-1)^{j+1} xT_j(x)S_j(x) = \frac{1}{2} + \begin{cases} \dfrac{T_{N+1}S_N}{2}, & N \text{ odd}, \\ \dfrac{-T_N S_{N+1}}{2}, & N \text{ even}. \end{cases} \tag{7.6*}$$

(B) $n = 0,\ m = 1$

$$\sum_{j=1}^{N} (-1)^{j+1}(j+1)A_j = a + \begin{cases} \dfrac{(N+1)A_{N+1}}{2}, & N \text{ odd}, \\ \dfrac{-(N+2)A_N}{2}, & N \text{ even}; \end{cases} \tag{7.1**}$$

$$\sum_{j=1}^{N} (q+1)q^{N-j}(-1)^{j+1}G_j^{(q)}U_{j+1} = q^N a(q+1) + \begin{cases} G_{N+1}^{(q)}U_{N+1}, & N \text{ odd}, \\ -G_N^{(q)}U_{N+2}, & N \text{ even}; \end{cases} \tag{7.2**}$$

$$\sum_{j=1}^{N} (-1)^{j+1}2^{N-j}M_jU_{j+1} = 2^{N+1} + \begin{cases} \dfrac{M_{N+1}U_{N+1}}{3}, & N \text{ odd}, \\ \dfrac{-M_NU_{N+2}}{3}, & N \text{ even}; \end{cases} \tag{7.3**}$$

$$\sum_{j=1}^{N} xF_j(x)P_{j+1}(x) = -x + \begin{cases} F_{N+1}(x)P_{N+1}(x), & N \text{ odd}, \\ F_N(x)P_{N+2}(x), & N \text{ even}; \end{cases} \tag{7.4**}$$

$$\sum_{j=1}^{N} xL_j(x)P_{j+1}(x) = -x^2 + \begin{cases} L_{N+1}(x)P_{N+1}(x) & N \text{ odd}, \\ L_N(x)P_{N+2}(x), & N \text{ even}; \end{cases} \tag{7.5**}$$

$$\sum_{j=1}^{N} (-1)^{j+1}xT_j(x)S_{j+1}(x) = x + \begin{cases} \dfrac{T_{N+1}(x)S_{N+1}(x)}{2}, & N \text{ odd}, \\ \dfrac{-T_N(x)S_{N+2}(x)}{2}, & N \text{ even}. \end{cases} \tag{7.6**}$$

Remark: Obviously, various other identities may be obtained by other choices of n and m. This bears out the fact that this technique provides an abundance of identities by substituting suitable values for m, n, p, and q is just one identity (6.1)!

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