# NEW UNITARY PERFECT NUMBERS HAVE AT LEAST NINE ODD COMPONENTS

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## 1. INTRODUCTION

We say that a divisor $d$ of an integer $n$ is a *unitary divisor* if

$$\gcd(d, n/d) = 1,$$

in which case we write $d \| n$. By a *component* of an integer we mean a prime power unitary divisor.

Let $\sigma^*(n)$ denote the sum of the unitary divisors of $n$. Then $\sigma^*$ is a multiplicative function, and $\sigma^*(p^e) = p^e + 1$ if $p$ is prime and $e \geqslant 1$. Throughout this paper we will let $f$ be the *ad hoc* function defined by $f(n) = \sigma^*(n)/n$.

An integer $n$ is *unitary perfect* if $\sigma^*(n) = 2n$, i.e., if $f(n) = 2$. Subbarao and Warren [2] found the first four unitary perfect numbers, and this author [3] found the fifth. No other such numbers have been found, so at this stage the only known unitary perfect numbers are:

$$6 = 2 \cdot 3,\ 60 = 2^2 3 \cdot 5;\ 90 = 2 \cdot 3^2 5;\ 87360 = 2^6 3 \cdot 5 \cdot 7 \cdot 13;\ \text{and}$$

$$146361946186458562560000 = 2^{18} 3 \cdot 5^4 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313$$

It is easy to show that any unitary perfect number must be even. Suppose that $N = 2^a m$ is unitary perfect, where $m$ is odd and $m$ has $b$ distinct prime divisors (i.e., suppose that $N$ has $b$ odd components). Subbarao and his co-workers [1] have shown that any new unitary perfect number $N = 2^a m$ must have $a > 10$ and $b > 6$. In this paper we establish the improved bound $b > 8$.

Much of this paper rests on a results in an earlier paper [4]:

> Any new unitary perfect number has an odd component larger than $2^{15}$ (the smallest candidate is 32771).

Essential to this paper is the ability to find bounds for the smallest unknown odd component of a unitary perfect number. The procedure is laborious but simple, and can be illustrated by an example:

Suppose $N = 2^a 3 \cdot 5 \cdot 7 \cdot 19 \cdot 43 \cdot rqp$ is unitary perfect, where $r$, $q$, and $p$ are distinct odd prime powers, $r < q < p$, $a \geqslant 12$, and $p \geqslant 32771$. Then $64 < r < 261$, because

$$f(3 \cdot 5 \cdot 7 \cdot 19 \cdot 43) \cdot (262/261)^4 < 2 < f(3 \cdot 5 \cdot 7 \cdot 19 \cdot 43) \cdot (65/64).$$

Consequently, $r < 2^a$ and $r < 32771$. But $f(2^a) \leqslant 4097/4096$ as $a \geqslant 12$, and

$$f(3 \cdot 5 \cdot 6 \cdot 19 \cdot 43) \cdot (4097/4096) \cdot (32772/32771) \cdot (134/132)^2 < 2,$$

so $64 < r < 133$.

In the interests of brevity, we will simply outline the proofs, omitting repetitive details.

## 2. SEVEN ODD COMPONENTS

Throughout this section, suppose $N = 2^a vutsrqp$ is unitary perfect, where $p, \ldots, v$ are powers of distinct odd primes, and $v < u < t < s < r < q < p$. Then we know that $a \geqslant 11$ and $p \geqslant 32771$.

**Theorem 2.1:** $v = 3$, $u = 5$, $t = 7$, and $a \geqslant 12$.

**Proof:** We have $v = 3$ or else $f(N) < 2$, so there is only one component $\equiv -1 \pmod 3$, and none $\equiv -1 \pmod 9$. But

$$f(2^{11} 3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 25 \cdot 32771) < 2,$$

so $u = 5$. Then there are no more components $\equiv -1 \pmod 3$, only one $\equiv -1 \pmod 5$, and none $\equiv -1 \pmod{25}$. As a result, $a$ is even, so $a \geqslant 12$. Then $t = 7$, or else $f(N) < 2$. ∎

**Theorem 2.2:** $s = 13$.

**Proof:** We easily have $s = 13$ or $s = 19$, or else $f(N) < 2$, so suppose $s = 19$. Then $25 < r < 53$. If $r$ is 43 or 37, then (respectively) $64 < q < 66$ or $85 < q < 88$, both of which are impossible. Thus, $r = 31$, so $151 < q < 159$ and then $q = 157$. But then $79 | p$ and $p > 2^{15}$, so $p = 79^c$ with $c \geqslant 3$, whence $79^2 | \sigma^*(2^a)$, which is impossible. ∎

**Theorem 2.3:** $r = 67$.

**Proof:** We have $N = 2^a 3 \cdot 5 \cdot 7 \cdot 13 \cdot rqp$, $p \geqslant 32771$, and $a \geqslant 12$, so $64 < r < 131$. If $r > 79$, easy contradictions follow.

If $r = 79$, then $341 < q < 377$, so $q = 361$, 367, or 373. But $q = 373$ implies $11 \cdot 17 | p$, a contradiction. If $q = 367$, then $p = 23^c$ with $c \geqslant 4$, so $23^3 | \sigma^*(2^a)$, which is impossible. If $q = 361$, then $p = 181^c$ with $c \geqslant 3$, so $181 | \sigma^*(2^a)$, hence $90 | a$, whence $5^2 | N$, a contradiction.

Finally, if $r = 73$, then $526 < q < 615$ and $37 | qp$, so $p = 37^c$ with $c \geqslant 3$. But $73 \nmid \sigma^*(2^a 37^c)$, so $73 | (q + 1)$, which is impossible.

**Theorem 2.4:** There is no unitary perfect number with exactly seven odd components.

**Proof:** If this is so, then $N = 2^a 3 \cdot 5 \cdot 7 \cdot 13 \cdot 67 \cdot qp$. Then $1450 < q < 4353$, so $p \geqslant 32771$, whence $1450 < q < 3037$. Then $a \geqslant 12$ implies $1450 < q < 2413$. Now, $17^3 \| N$ implies $3^3 | N$, so $p = 17^c$ with $c \geqslant 4$. But $17^2 \nmid \sigma^*(2^a)$, or else $q$ is a multiple of 354689, so $17^3 | (q + 1)$, which is impossible. ■

## 3. EIGHT ODD COMPONENTS

Throughout this section, assume that $N = 2^a wvutsrqp$ is unitary perfect, where $p, \ldots, w$ are powers of distinct odd primes, and $w < v < u < t < s < r < q < p$. Then $a \geqslant 11$ and $p \geqslant 32771$ as before.

**Theorem 3.1:** $w = 3$, $v = 5$, and $a \geqslant 12$.

**Proof:** Similar to that for Theorem 2.1. ■

**Theorem 3.2:** $u = 7$, and $t = 13$ or $t = 19$.

**Proof:** From $f(2^{12} 3 \cdot 5 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 43 \cdot 32771) < 2$, we have $u = 7$, so there is only one component $\equiv -1 \pmod 7$. Thus, $t \leqslant 31$. If $t$ is neither 13 nor 19, then $t = 31$, so $a \geqslant 14$, and we quickly obtain $s = 37$ and $r = 43$. But then we have $N = 2^a 3 \cdot 5 \cdot 7 \cdot 31 \cdot 37 \cdot 43 \cdot qp$, subject to $121 < q < 125$ and $11 \cdot 19 | qp$, an impossibility. ■

**Theorem 3.3:** If $t = 19$, then $s = 31$.

**Proof:** Suppose $N = 2^a 3 \cdot 5 \cdot 7 \cdot 19 \cdot srqp$ with $s < r < q < p$. Then $25 < s < 73$. Easy contradictions follow if $s > 43$.

If $s = 43$, then $64 < r < 133$. If $r = 121$, then $140 < q < 147$, which is impossible. Other choices for $r$ force $q$ and $p$ to be powers of 11 and another odd prime (in some order) with no acceptable choice for $q$ in its implied interval.

If $s = 37$, then $85 < r < 176$, so $r$ is 103, 121, 127, 157, or 163. If $r$ is 157 or 163, there in only one choice for $q$, and it implies that $p$ is divisible by two different odd primes. If $r = 127$, then $a > 20$ and so $262 < q < 265$, an impossibility. If $r = 121$, then $291 < q < 318$, so $q$ is 307 or 313; but $q = 313$ implies $61 \cdot 157 | p$, and if $q = 307$, then $p = 61^c$ with $c \geqslant 3$, so $61^2 | \sigma^*(2^a)$, whence $5^2 | N$, a contradiction. If $r = 103$, then $502 < q < 583$ and $13 | qp$, so $p = 13$ with $c \geqslant 4$; but $13 \nmid \sigma^*(2^a)$, or else $5^2 | N$, so $13^3 | (q + 1)$, which is impossible. ■

**Theorem 3.4:** $t = 13$.

**Proof:** If $t \neq 13$, then $N = 2^a 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot rqp$ with $r < q < p$ and $a \geqslant 16$, so $151 < r < 307$. Since $r \not\equiv -1 \pmod 5$, $r$ must be 157, 163, 181, 193, 211, 223, 241, 271, 277, or 283. If $r$ is 271, 241, or 223, there is no prime power in the implied interval for $q$ (note $a \geqslant 20$ if $r = 223$). If $r$ is 283, 277, 211, or 193, the only choices for $q$ require that $p$ be divisible by two distinct primes.

If $r = 163$, then $2202 < q < 2450$, so $p = 41$ with $c \geqslant 4$; thus, $2202 < q < 2281$, and the only primes that can divide $q + 1$ are 2, 7, 19, 31, 41, and 163, but no such $q$ exists. If $r = 181$, then $p = 13^c$ with $c \geqslant 4$, as $942 < q < 985$ and $13 | qp$; but $13 \nmid \sigma^*(2^a)$, or else $5^2 | N$, so $13^3 | (q + 1)$, which is impossible. If $r = 157$, then $79 | qp$ and $4525 < q < 5709$, so $p = 79^c$ with $c \geqslant 3$; however, $79 \nmid \sigma^*(2^a)$, and so $79^2 | (q + 1)$, an impossibility. ■

**Corollary:** There are no more components $\equiv -1 \pmod 7$, and none $\equiv -1 \pmod{13^2}$.

**Theorem 3.5:** $s \leqslant 73$.

**Proof:** We have $N = 2^a 3 \cdot 5 \cdot 7 \cdot 13 \cdot srqp$, and $61 < s < 193$ follows easily, so $s$ is 67, 73, 79, 103, 109, 121, 151, 157, or 163.

If $s$ is 163 or 157, then any acceptable choice of $r$ forces $qp$ to be divisible by two distinct odd primes with no acceptable choice for $q$ in its implied interval. The same occurs with $s = 151$ unless $r = 163$; but if $s = 151$ and $r = 163$, then $358 < q < 398$ and $19 \cdot 41 | qp$, so $q = 19^2$, whence $41 \cdot 181 | p$, an impossibility. If $s = 127$, then $a \geqslant 16$ and, for each $r$, any acceptable choice for $q$ forces $p$ to be divisible by two distinct primes.

If $s = 121$ and $r \neq 241$, then two known odd primes divide $qp$ and there is no acceptable choice for $q$ in its implied interval. If $s = 121$ and $r = 241$, then $318 < q < 350$ and $61 | qp$, so $p = 61^c$ with $c \geqslant 3$; but $61 \nmid \sigma^*(2^a)$ unless $41 | q$, hence $61^2 | (q + 1)$, which is impossible.

Suppose $s = 109$. Then $156 < r < 328$ and $11 | rqp$, so $11^4 | qp$ as $11^3 \| N$ implies $3^2 | N$. Now, $109 \nmid \sigma^*(2^a)$, or else $5^2 | N$. If $109 | \sigma^*(11^c)$, then $11 \cdot 61 \cdot 1117 | rqp$, an impossibility. Thus, one of $q$ and $p$ is $11^c$ with $c \geqslant 4$, and the other is a component $\equiv -1 \pmod{109}$, and the least candidate for this component is 2833. Then $156 < r < 175$, so $r$ is 157 or 163. If $r = 163$, then $a \neq 12$, or else $11 \cdot 17 \cdot 41 \cdot 241 | rqp$, so $a \geqslant 14$, whence $11 \cdot 41 | qp$ and $3913 < p < 6100$, an impossibility. If $r = 157$, then $a \geqslant 16$, and $11 \cdot 79 | qp$ and $44000 < q < 300000$, whence $q = 11^5$ and $3^2 | N$, a contradiction.

If $s = 103$ and $r = 271$, then $a \geqslant 16$ and $462 < q < 473$, so $q = 463$ and $17 \cdot 29 | p$, an impossibility. If $s = 103$ and $r \neq 271$, then $r + 1$ includes an odd prime $\pi$ and the interval for $q$ forces $p = \pi^c$ $(c \geqslant 2)$. But in each case, $\pi | \sigma^*(2^a)$ implies a contradiction, so $\pi^{c-1} | (q + 1)$, an impossibility.

If $s = 79$, then $a \geqslant 16$, as $a = 14$ implies $5^2 | N$, so $341 < r < 695$. Except for $r = 373$, $r + 1$ includes an odd prime $\pi$ and the interval for $q$ forces $p = \pi^c$

($c \geqslant 2$), but in each instance $\pi | \sigma^*(2^a)$ either is impossible or implies conditions on $q$ which cannot be met. If $r = 373$, then $4031 < q < 4944$ and $11 \cdot 17 | qp$, so $q = 17^3$, whence $3^2 | N$, a contradiction. ∎

**Theorem 3.6:** $s = 67$.

**Proof:** Suppose not: then $N = 2^a 3 \cdot 5 \cdot 7 \cdot 13 \cdot 73 \cdot rqp$, $526 < r < 1232$, and $37 | rqp$. The cases $37^2 \| N$ and $37^3 \| N$ are easily eliminated, so $37^4 | N$. Now, $73 \nmid \sigma^*(2^a 37^c)$, so $N$ has an odd component, not $37^c$, which is $\equiv -1 \pmod{73}$, and the two smallest candidates are 1459 and 5839. If $N = 2^a 3 \cdot 5 \cdot 7 \cdot 13 \cdot 73 \cdot 1459 \cdot qp$, then $823 < q < 1032$, but $37 \nmid \sigma^*(2^a)$, or else $5^2 | N$, so $37^3 | (q+1)$, which is impossible.

Now, call $p = 37^c$ ($c \geqslant 4$), $q \equiv -1 \pmod{73}$, and $q \geqslant 5839$. Then $526 < r < 674$, so $37 \nmid (r+1)$. Consequently, $q \equiv -1 \pmod{37^3}$, so $q + 1 \geqslant 2 \cdot 37^3 73$ and, hence, $q \geqslant 7395337$. If $a = 12$ or $a = 14$, then $r$ is in an interval with no prime powers. Therefore, $a \geqslant 16$, so $526 < r < 531$, which forces $r = 529$. Then $a \geqslant 18$, but $a = 18$ implies $5^2 | N$, so $a \geqslant 20$. But then $100000 < q < 240000$ and $53 \cdot 37 | qp$, so $q = 53^3$, which implies $3^2 | N$, a contradiction. ∎

**Theorem 3.7:** There is no unitary perfect number with exactly eight odd components.

**Proof:** Assume not: then we have $N = 2^a 3 \cdot 5 \cdot 7 \cdot 13 \cdot 67 \cdot rqp$ with $1450 < r < 4825$. Now, $67 \nmid \sigma^*(2^a)$, or else $3^2 | N$. Also, $17 | N$ and $17^2 < r$. But 17 cannot divide $N$ an odd number of times, or else $3^2 | N$, so $17^4 | N$.

We already have $a \geqslant 12$ and $a$ even. The cases $a = 12$ and $a = 14$ are easily eliminated, so $a \geqslant 16$ and then $1450 < r < 3022$.

Note that $67 \nmid \sigma^*(17^c)$, so $N$ has an odd component, not $17^c$, which is $\equiv -1 \pmod{67}$, and the three smallest candidates are 1741, 2143, and 4153. If the component $\equiv -1 \pmod{67}$ exceeds 2143, then $1450 < r < 2375$. Thus, we may require $1450 < r < 2375$ in any event.

We cannot have $17^2 | \sigma^*(2^a)$, or else $17 \cdot 3546898 \cdot 2879347902817 | rqp$, and this is obviously impossible. If $17 | (r+1)$, then $r$ is 1597, 1801, 2209, or 2311. If $67 | (r+1)$, then $r$ is 1741 or 2143. If $r + 1$ is divisible by neither 17 nor 67, then we may take $p = 17^c$ ($c \geqslant 4$, so $p \geqslant 83521$) and $q \equiv -1 \pmod{17^2 67}$, whence $q \geqslant 116177$, so $1450 < r < 1531$. Thus, in any event, $r$ must be one of the following numbers: 1453, 1459, 1471, 1489, 1597, 1741, 1801, 2143, 2209, or 2311. But each of these cases leads to a contradiction, so the theorem is proved. ∎

## REFERENCES


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