OPTIMAL SPACING OF POINTS ON A CIRCLE

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1. Introduction

Consider N points placed on the circle of unit circumference in the following way: begin by placing a point anywhere on the circle. Now place another point so that the angle (or circumferential distance) between the two points, measured clockwise from the first point, is equal to α. The third point is now placed at a clockwise angle of α from the second point. Thus, we successively place N points on the circle by our angle α.

Our problem is to find the value for α so that these points are spread about the circle in the most even (which we call optimal) fashion. We show that, in certain senses, the golden section ($\alpha = \tau = (\sqrt{5} - 1)/2$) provides the optimal spacing of points, where the number of points can assume any value.

This problem originally arose while investigating the phenomenon of phyllotaxis—regular leaf arrangement. Most higher-order plants exhibit a remarkable degree of regularity in the positioning of their leaves. In a sunflower, for instance, one can perceive two sets of opposed spirals which each partition the set of florets. Intriguingly, the number of spirals are almost certainly consecutive members of the Fibonacci sequence

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2,\ F_0 = 0,\ F_1 = 1.$$

This pattern (which we call Fibonacci phyllotaxis) manifests itself in 95% of those plants which produce their leaves sequentially. In parallel to this observation, the divergence angle subtended by consecutively formed leaves is quite close in value to the ratio of these consecutive Fibonacci numbers. In the limit, F_{n-1}/F_n is equal to the golden section. To simplify the situation, we consider just the angular displacement of the leaves and thus we develop a simplistic model of plant growth with leaves appearing as points on a meristematic ring, successively placed at a constant angle.

What this paper shows is that the plant places its leaves in the optimal manner—in order to spread its leaves most evenly (and thus reduce leaf overlap) the optimal divergence angle is shown to be the golden section. The partition of the circle by the golden section is also examined in detail to reveal a rather self-similar structure.

We use results from The Three Gap Theorem (originally the Steinhaus Conjecture) which states that the above N points partition the circle into arcs, or gaps, of at most three and at least two different lengths! The result is all the more remarkable since it holds for all irrational α *and* for any number of points. It also holds for rational $\alpha = p/q$ with the number of points less than q. (For $N = q$ the circle is partitioned into q equal gaps.) Even though this has been proved by various mathematicians ([1], [2]-[7]), the result does not appear to be well known.

Note that, in order to conserve space, where complete proofs of results are not presented we either refer the reader to an existing proof or briefly outline a proof.

2. The Three Gap Theorem

Suppose that we have consecutively placed N points on a circle by the angle α. Let $(u_1(N), u_2(N), \ldots, u_N(N))$ be the sequence of points as they appear on the circle, ordered clockwise from the origin $u_1(N) = 0$. That is,

$$\{u_1(N), u_2(N), \ldots, u_N(N)\} = \{0, 1, 2, \ldots, N - 1\} \text{ where } \{u_j\alpha\} < \{u_{j+1}\alpha\}.$$

Thus, for example, with $\alpha = \sqrt{2}$ the first 12 points placed on the circle appear in the order (0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7). We call $u_{j+1}(N) = u_{j+1}$ the successor to u_j, or $u_{j+1} = \mathrm{Suc}(u_j)$. Equivalent to the original statement of The Three Gap Theorem is the fact that the difference between succeeding points assumes at most three, and at least two, different values. The following determines the ordering of points around the circle. (For a proof, see van Ravenstein [7, Theorem 2.2].).

Theorem 1:

$$\mathrm{Suc}(m) - m = \begin{cases} u_2, & 0 \le m < N - u_2, \\ u_2 - u_N, & N - u_2 \le m < u_N, \\ -u_N, & u_N \le m < N. \end{cases}$$

Thus, for our example with $\alpha = \sqrt{2}$ and $N = 12$,

$$\mathrm{Suc}(m) - m = \begin{cases} 5, & 0 \le m < 7, \\ -7, & 7 \le m < 12. \end{cases}$$

It is easily seen that $u_j = 5(j - 1) \bmod 12$, where $y \bmod x = y - x[y/x] = x\{y/x\}$. In general, if $N = u_2 + u_N$, the circle is partitioned into gaps of just two different lengths and then

$$u_j = ((j - 1)u_2) \bmod N, \; j = 1, 2, \ldots, N. \tag{1}$$

It is easy to see that the length of the gap formed by point m and $\mathrm{Suc}(m)$ is equal to $\{(\mathrm{Suc}(m) - m)\alpha\}$ where $\{x\}$ denotes the fractional part of x such that $x = [x] + \{x\}$ where $[x]$ is the largest integer not greater than x. In fact, for gap lengths less than $\frac{1}{2}$ this gap length is equal to

$$\|(\mathrm{Suc}(m) - m)\alpha\|, \quad \text{where } \|x\| = \min(\{x\}, 1 - \{x\}) = |x - [x + \tfrac{1}{2}]|,$$

the difference between x and its nearest integer. (This is always the case for $N > q_1$ [notation defined in Theorem 2]; in what follows, we will always make this assumption. Note that q_1 is the first point to replace 1 as the closest point to the origin.) Thus, Theorem 1 shows that the circle of N points is partitioned into $N - u_2$ gaps of length $\|u_2\alpha\|$, $N - u_N$ gaps of length $\|u_N\alpha\|$ and $u_2 + u_N - N$ gaps of length $\|u_2\alpha\| + \|u_N\alpha\|$. The same applies for rational α, say $\alpha = p/q$ in lowest terms, where $N < q$. In this paper, however, we will always assume that α is irrational.

Point u_2 is the successor to 0, while 0 is the successor to u_N; that is, u_2 and u_N are the points which neighbor the origin. We see that we need only know the values of these two points to determine the entire ordering.

We can characterize the angle α by the following. Let $V(\alpha)$ denote the *path* of α defined to be a sequence of pairs (u_2, u_N), the points which neighbor the origin as points are successively included on the circle. For example,

$$V(\sqrt{2}) = ((1, 2), (3, 2), (5, 2), (5, 7), (5, 12), \ldots).$$

It can be shown ([7, Proposition 4.2]) that each point always enters one of the larger gaps. Two gaps are formed, one equal in length to the smallest gap present. Thus, it is natural to define the ratio of gap division as the ratio of the smallest to the largest gap present. Hence, we let

$$r_N(\alpha) = \frac{\min(\|u_2\alpha\|, \|u_N\alpha\|)}{\|u_2\alpha\| + \|u_N\alpha\|}.$$

[This is in fact the ratio point $N - 1$ that divides some (large) gap.]

The path sequence $V(\alpha)$ and the ratio of gap division $r_N(\alpha)$ are quantities we will use in our analysis of the golden section's unique distribution properties. We can in fact determine explicitly their values in terms of the continued fraction expansion of α, which is expressed by

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

$$\{a_0;\ a_1,\ a_2,\ a_3,\ \ldots\}.$$

The n^{th} tail of α is

$$t_n = \{a_n;\ a_{n+1},\ a_{n+2},\ \ldots\}, \tag{2}$$

such that

$$\alpha = \{a_0;\ a_1,\ a_2,\ \ldots,\ a_{n-1},\ t_n\}.$$

We say that α is *equivalent* to β if some tail in α is equal to some tail in β.

Partial convergents are defined by the (irreducible) fractions

$$\frac{p_{n,i}}{q_{n,i}} = \frac{p_{n-2} + ip_{n-1}}{q_{n-2} + iq_{n-1}} = \{a_0;\ a_1,\ a_2,\ \ldots,\ a_{n-1},\ i\},\ i = 1,\ 2,\ \ldots,\ a_n,$$

where

$$\frac{p_{n,a_n}}{q_{n,a_n}} = \frac{p_n}{q_n},\ p_{-2} = q_{-1} = 0,\ q_{-2} = p_{-1} = 1.$$

We call p_n/q_n a *total* convergent to α.

The reader is referred to [7, Theorem 3.3] for a proof of the following.

Theorem 2:

$$u_2 = \begin{cases} q_{n-1}, & n \text{ odd}, \\ q_{n,i-1}, & n \text{ even}, \end{cases} \qquad u_N = \begin{cases} q_{n,i-1}, & n \text{ odd}, \\ q_{n-1}, & n \text{ even}, \end{cases}$$

where $q_{n,i-1} < N \le q_{n,i}$, $2 \le i \le a_n$ $(n \ge 2)$.

For $q_{n-1} < N \le q_{n,1}$ $(n \ge 2)$,

$$u_2 = \begin{cases} q_{n-1}, & n \text{ odd}, \\ q_{n-2}, & n \text{ even}, \end{cases} \qquad u_N = \begin{cases} q_{n-2}, & n \text{ odd}, \\ q_{n-1}, & n \text{ even}. \end{cases}$$

For $N \le q_1$, $u_j = j - 1$, $j = 1,\ 2,\ \ldots,\ N$.

The following proposition may be easily proved from the definition of $r_N(\alpha)$, Theorem 2, and the continued fraction theory.

Proposition 3:

$$r_N(\alpha) = \begin{cases} \dfrac{1}{1 + t_n}, & q_{n-1} < N \le q_{n,1}, \\[2ex] \dfrac{1}{2 - i + t_n}, & q_{n,i-1} < N \le q_{n,i}, \end{cases}$$

where $i = 2, 3, \ldots, a_n$, $(n \geq 2)$. [t_n is defined by (2).]

From Theorem 2,

$$V(\alpha) = ((1, q_1)', (q_{n,i}, q_{n-1})''; i = 1, 2, \ldots, a_n, n = 2, 3, \ldots), \quad (3)$$

where

$$(1, q_1)' = \begin{cases} (1, q_1), & 0 < \alpha < \frac{1}{2}, \\ (q_1, 1), & \frac{1}{2} < \alpha < 1, \end{cases}$$

$$(q_{n,i}, q_{n-1})'' = \begin{cases} (q_{n-1}, q_{n,i}), & n \text{ odd}, \\ (q_{n,i}, q_{n-1}), & n \text{ even}. \end{cases}$$

3. The Golden Section

For convenience, let the partition of the circle of unit circumference by the successive placement of points 0, 1, 2, ... by the golden section, τ, be denoted by G. The partition by $1 - \tau$ we denote by G'.

The continued fraction of τ is given by

$$\tau = \{0; 1 + \tau\} = \{0; 1, 1 + \tau\} = \{0; 1, 1, 1, \ldots\}.$$

All convergents to τ are total convengents and

$$p_n = q_{n-1} = F_n = F_{n-1} + F_{n-2}, \; n \geq 1, \; F_{-1} = 1, \; F_0 = 0.$$

That is, convergents to τ are equal to the ratio of consecutive Fibonacci numbers. From Theorems 1 and 2, for $F_n < N \leq F_{n+1}$,

$$\operatorname{Suc}(m) - m = \begin{cases} \left.\begin{array}{ll} F_n, & 0 \leq m \leq N - F_n, \\ F_{n-2}, & N - F_n < m < F_{n-1}, \\ -F_{n-1}, & F_{n-1} \leq m < N, \end{array}\right\} & n \text{ odd}, \\ \left.\begin{array}{ll} F_{n-1}, & 0 \leq m \leq N - F_{n-1}, \\ -F_{n-2}, & N - F_{n-1} < m < F_n, \\ -F_n, & F_n \leq m < N, \end{array}\right\} & n \text{ even}. \end{cases} \quad (4)$$

When $N = F_{n+1}$, from (1),

$$u_j = ((-1)^{n-1}(j - 1)F_n) \bmod F_{n+1}.$$

Since $F_{n-1} - F_n\tau = (-\tau)^n$ (by induction), (4) shows that N points ($F_n < N \leq F_{n+1}$) partition the circle into $N - F_{n-1}$ gaps of length τ^{n-1}, $N - F_n$ gaps of length τ^n and $F_{n+1} - N$ gaps of length τ^{n-2}.

From Proposition 3, since $t_n = 1 + \tau$, $n = 1, 2, \ldots$,

$$r_N(\tau) = \tau^2 = 1 - \tau, \quad F_n < N \leq F_{n+1}. \quad (5)$$

Theorem 4.1 from [7] describes the partition G by looking at the transformation of gap types as points are included on the circle. Gap types are either "large" or "small" when $N = u_2 + u_N$, that is, when N is the denominator of a convergent to α. For the golden section, this theorem describes the following: each large gap present when $N = F_n$ is divided by the addition of a further F_{n-1} points into two new gaps which can be labelled (in clockwise order) as small:large (n odd) or large:small (n even) when $N = F_{n+1}$. Those small gaps present when $N = F_n$ can then be labelled as large.

This fact and (5) can be used to prove the following self-similarity property of G, thus demonstrating the beautiful symmetry inherent in Fibonacci phyllotaxis.

Theorem 4: Consider any large gap present on the circle partitioned by the placement of F_n points by the golden section. Include further points on the circle and observe the resulting partitioning of this gap. If we pretend the gap is itself a circle of unit circumference (by lengthening it by the factor τ^{n-2} and identifying its endpoints as the same) then its partition is identical to G if n is odd, or equal to G' if n is even.

Let us interpret N to be a time variable and define the age of a gap to be the time it has survived without being divided. That is, the age of the gap with endpoints u_j, u_{j+1} is $N - 1 - \max(u_j, u_{j+1})$. From [7, Proposition 4.2] each point, for all α, divides the oldest of the larger gaps. Using [7, Theorem 4.1] it can be shown that only for the golden section does the formation of a large gap always coincide with that of a small gap. This proves the following. (Note that we assume that α is between 0 and 1. If $\alpha > 1$, the following results hold if α is replaced by its fractional part, $\{\alpha\}$.)

Theorem 5: For $\frac{1}{2} < \alpha < 1$, each point always enters the oldest gap if and only if $\alpha = \tau$. For $0 < \alpha < \frac{1}{2}$, each point enters the oldest gap if and only if $\alpha = \tau^2 = (3 - \sqrt{5})/2$.

Intuitively, in terms of phyllotaxis, it seems sensible that points be inserted in the oldest gap as the above result shows. This property must ensure an ideal distribution of points. In fact, the following theorem shows that the golden section provides the optimal value for gap division (our criteria for an optimal distribution) in the sense that the smallest value assumed by the ratio of gap division is *largest* for the golden section (where $\frac{1}{2} < \alpha < 1$). However, the golden section is somewhat of a compromise as Theorem 7a shows (that the ratio of gap division's maximum value is smallest for the golden section, where $\frac{1}{2} < \alpha < \frac{2}{3}$).

Theorem 6: $\displaystyle\max_{\frac{1}{2}<\alpha<1} \min_N r_N(\alpha) = \tau^2$, exclusively attained by $\alpha = \tau$.

$\displaystyle\max_{0<\alpha<\frac{1}{2}} \min_N r_N(\alpha) = \tau^2$, exclusively attained by $\alpha = \tau^2$.

Proof: We first consider the case where $\frac{1}{2} < \alpha < 1$. From Proposition 3,

$$\min_N r_N(\alpha) = \min_n \min_{q_{n-1} < N \le q_n} r_N(\alpha) = \min_n \frac{1}{1 + t_n} = \frac{1}{1 + \max_n t_n},$$

where $n = 2, 3, \ldots$ ($q_1 = a_1 = 1$ since $\alpha > \frac{1}{2}$).

Consider $\alpha = \alpha' \neq \tau$, which has $a_k > 1$ for some integer k greater than 1. Then $\max_n t_n \geq t_k > 2$, so $r_N(\alpha') < \frac{1}{3}$. The result follows since $\frac{1}{3} < r_N(\tau) = \tau^2$.

The second statement follows by symmetry (note that $r_N(1 - \alpha) = r_N(\alpha)$).

Theorems 7a and 7b follow from Proposition 3 in a similar fashion.

Theorem 7a: $\displaystyle\min_{\frac{1}{2}<\alpha<\frac{2}{3}} \max_N r_N(\alpha) = \tau^2$, exclusively attained by $\alpha = \tau$.

$\displaystyle\min_{\frac{1}{3}<\alpha<\frac{1}{2}} \max_N r_N(\alpha) = \tau^2$, exclusively attained by $\alpha = \tau^2$.

Theorem 7b: $\displaystyle\min_{\frac{2}{3}<\alpha<1} \max_N r_N(\alpha) = \tau^2$, exclusively attained by

$$\alpha = \{0; 1, a, 1 + \tau\} = \frac{\tau + a}{\tau + a + 1},$$ where a is any integer greater than 1.

$$\min_{0<\alpha<\frac{1}{3}} \max_N r_N(\alpha) = \tau^2, \text{ exclusively attained by}$$

$$\alpha = \{0;\, a+1,\, 1+\tau\} = \frac{1}{\tau + a + 1}, \text{ where } a \text{ is any integer greater than 1.}$$

We determine the value of α which ensures a path which consistently maximizes the length of the smallest gap on the circle. For each value of N, the points are generated by a constant angle α. This value of α may change with N but only in such a way that the path is retained: so that the addition of extra points does not alter the relative order of existing points. We show that $V(\tau)$ is the path which ensures that the smallest gaps are consistently as large as possible [where, initially, $(u_2(4),\, u_3(4)) = (2,\, 1)$, $\frac{1}{2} < \alpha < \frac{2}{3}$]. That is, as N increases, if the pair $(u_2(N),\, u_N(N))$ does not assume the value equal to the appropriate successive element of $V(\tau)$, then the smallest gap thus formed will not be as large.

Note that the golden section has path

$$\begin{aligned} V(\tau) &= ((1,\, 1),\, (2,\, 1),\, (2,\, 3),\, (5,\, 3),\, \ldots), \\ &= ((1,\, 1),\, (F_{n+1},\, F_n)',\, n = 2,\, 3,\, \ldots). \end{aligned}$$

Theorem 8: Suppose that $(u_2(4),\, u_3(4)) = (2,\, 1)$ generated by a constant angle α where $\frac{1}{2} < \alpha < \frac{2}{3}$. Then $V(\tau)$ is the path which consistently maximizes the length of the smallest gap.

Proof: We prove the result by induction. Initially, $\frac{1}{2} < \alpha < \frac{2}{3}$ or $\alpha = \{0;\, 1,\, 1,\, t_3\}$, $1 < t_3 < \infty$, such that point 2 is closest to the origin. The next element in the path must, from (3), be (2, 3). From Proposition 3, point 2 is furthest from the origin if $a_3 = 1$ than if $a_3 > 1$ since then it divides the gap bordered by the origin and the first point into a larger ratio. Hence, $\alpha = \{0;\, 1,\, 1,\, 1,\, t_4\}$. This ensures, from (3), that the next element in the path is (5, 3). Thus, the first three terms in the path belong to $V(\tau)$.

Now, assume that the terms in the path equal successive Fibonacci pairs and that $(u_2(N),\, u_N(N)) = (F_{n-1},\, F_n)$ where $F_n < N \le F_{n+1}$, n even. Then, $\alpha = \{0;\, 1,\, 1,\, 1,\, \ldots,\, 1,\, t_n\}$ $(n - 1)$ ones, $1 < t_n < \infty$. The next element in the path must be $(F_{n+1},\, F_n)$ succeeded by $(F_{n+1},\, F_{n+2})$ if $a_n = 1$. From Proposition 3, the small gap bordered by origin and point F_n is larger if $a_n = 1$ than if $a_n > 1$. The case is similar for odd n. Thus, the path is equal to $V(\tau)$.

Note that the theorem shows that maximizing the length of the smallest gap ensures convergence to the golden section. Similarly, $V(\tau^2)$ consistently maximizes the length of the smallest gap where, initially, there are three points on the circle and $\frac{1}{3} < \alpha < \frac{1}{2}$. The following generalizes Theorem 8. Its proof is similar in manner and is omitted.

Theorem 9: Suppose that we have placed $q_n + 1$ points $(n \ge 2)$ generated by $\alpha = \{0;\, a_1,\, a_2,\, \ldots,\, a_n,\, t_{n+1}\}$. Then as more points are added, $V(\alpha')$, where

$$\alpha' = \{0;\, a_1,\, a_2,\, \ldots,\, a_n,\, 1+\tau\} = \frac{p_n + \tau p_{n-1}}{q_n + \tau q_{n-1}},$$

is the path which consistently maximizes the length of the smallest gap.

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