ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-430 *Proposed by Larry Taylor, Rego Park, NY (Corrected Version)*

Find integers j, k ($\neq 0, \pm 1, \pm 2$), m_i and n_i such that:

(A) $5F_{m_i}F_{n_i} = L_k + L_{j+i}$, for $i = 1, 5, 9, 13, 17, 21$;

(B) $5F_{m_i}F_{n_i} = L_k - L_{j+i}$, for $i = 3, 7, 11, 15, 19, 23$;

(C) $F_{m_i}L_{n_i} = F_k + F_{j+i}$, for $i = 1, 2, \ldots, 22, 23$;

(D) $L_{m_i}F_{n_i} = F_k - F_{j+i}$, for $i = 1, 3, \ldots, 21, 23$;

(E) $L_{m_i}L_{n_i} = L_k - L_{j+i}$, for $i = 1, 5, 9, 13, 17, 21$;

(F) $L_{m_i}L_{n_i} = L_{-k} + L_{j+i}$, for $i = 2, 4, 6, 8$;

(G) $L_{m_i}L_{n_i} = L_k + L_{j+i}$, for $i = 3, 7, 11, 15, 16, 18, 19, 20, 22, 23$;

(H) $L_{m_i}L_{n_i} = L_k + F_{j+i}$, for $i = 10$;

(I) $L_{m_i}F_{n_i} = L_k + F_{j+i}$, for $i = 12$;

(J) $5F_{m_i}F_{n_i} = L_k + F_{j+i}$, for $i = 14$.

H-433 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let P_0, P_1, ... be the Pell numbers defined by

$$P_0 = 0,\ P_1 = 1,\ P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geq 2.$$

Show that, for $n = 1, 2, \ldots$,

$$6(n+1)P_{n-1} + P_{n+1} \equiv (-1)^{n+1}(9n^2 - 7)F_{n+1} \pmod{27}.$$

H-434 *Proposed by Piero Filipponi & Odoardo Brugia, Rome, Italy*

Strange creatures live on a planet orbiting around a star in a remote galaxy. Such beings have three sexes (namely, sex A, sex B, and sex C) and are reproduced as follows:

(i) An individual of sex A (or simply A) generates individuals of sex C by parthenogenesis.

(ii) If A is fertilized by an individual of sex B, then A generates individuals of sex B.

(iii) In order to generate individuals of sex A, A must be fertilized by both an individual of sex B and an individual of sex C.

Find a closed form expression for the number T_n of ancestors of an individual of sex A in the n^{th} generation. Note that, according to (i), (ii), and (iii), A has three parents ($T_1 = 3$) and six grandparents ($T_2 = 6$).

SOLUTIONS

A Prize Problem

H-409 *Proposed by John Turner, University of Waikato, New Zealand (Vol. 25, no. 2, May 1987)*

The following arithmetic triangle has many properties of special interest to Fibonacci enthusiasts.

							1							
						1	1	1						
					1	2	2	2	1					
				1	3	4	5	4	3	1				
			1	4	7	10	11	10	7	4	1			
		1	5	11	18	24	26	24	18	11	5	1		
	1	6	16	30	46	58	63	58	46	30	16	6	1	
1	7	22	47	81	116	143	153	143	116	18	47	22	7	1

....

Denote the triangle by T, the i^{th} element in the n^{th} row by t_i^n, and the sum of elements in the n^{th} row by σ_n.

(i) Discover a rule to generate the next row from the previous rows.

(ii) Given your rule, prove the Fibonacci row-sum property, viz:

$$\sigma_n = 2\sum_{i=1}^{n-1} t_i^n + t_n^n = F_{2n}, \text{ for } n = 1, 2, \ldots,$$

where F_{2n} is a Fibonacci integer.

(iii) Discover and prove a remarkable functional property of the sequence of diagonal sequences, $\{d_i\}$:

$d_1 = 1 \quad 1 \quad 1 \quad 1 \quad 1 \ldots$

$d_2 = 1 \quad 2 \quad 3 \quad 4 \quad 5 \ldots$

$d_3 = 1 \quad 2 \quad 4 \quad 7 \quad 11 \ldots$

$d_4 = 2 \quad 5 \quad 10 \quad 18 \quad 30 \ldots$

$d_5 = 1 \quad 4 \quad 11 \quad 24 \quad 46 \ldots$

(iv) Discover another Fibonacci arithmetic triangle which has the same generating rule and other properties but with row-sums equal to F_{2n-1}, $n = 1, 2, \ldots$.

(v) Show how the numbers in the triangle are related to the dual-Zeckendorf theorem on integer representations, which states (see [1]) that every positive integer N has one and only one representation in the form

$$N = \sum_{i}^{k} e_i u_i,$$

where the e_i are binary digits, $e_i + e_{i+1} \neq 0$ for $1 \leq i < k$, and $\{u_i\} = 1, 2, 3, 5, \ldots$, the Fibonacci integers.

There are many interesting identities derivable from the triangle, relating the t_i^n with themselves, with the natural numbers and Fibonacci integers, and with the binomial coefficients. The proposer offers a prize of US$25 for the best list of identities submitted.

A final remark is that Pascal-T and Fibonacci-T triangles (see [2] and [3]) can curiously be linked to a common source. They both may be derived from studies of binary words whose digits have the properties of the e_i in (v) above.

References

1. J. L. Brown, Jr. "A New Characterization of the Fibonacci Numbers." *Fibonacci Quarterly 3.1* (1965):1-8.
2. S. J. Turner. "Probability via the N^{th} Order Fibonacci-T Sequence." *Fibonacci Quarterly 17.1* (1979):23-28.
3. J. C. Turner. "Convolution Trees and Pascal-T Triangles." *Fibonacci Quarterly 26.4* (1988):354-365.

Solution by Karl Dilcher, Halifax, Nova Scotia

(i) Claim: Each element in the n^{th} row of T is the sum of the three closest elements in the $(n-1)^{\text{th}}$ row minus the closest element in the $(n-2)^{\text{th}}$ row.

Proof: Let

$$G(z, t) := \frac{1}{1 - t(1 + z + z^2) + z^2 t^2} = \sum_{n=0}^{\infty} f_n(z) t^n. \tag{1}$$

The $f_n(z)$ are polynomials of degree $2n$, and we have the recursion

$$\begin{aligned} &f_0(z) = 1,\ f_1(z) = 1 + z + z^2, \text{ and} \\ &f_{n+1}(z) = (1 + z + z^2) f_n(z) - z^2 f_{n-1}(z). \end{aligned} \tag{2}$$

The $f_n(z)$ are self-inverse polynomials, i.e., $f_n(z) = z^{2n} f_n(1/z)$; hence, we can write

$$\begin{aligned} f_n(z) = t_1^{n+1} + t_2^{n+1} z + \cdots + t_n^{n+1} z^{n-1} + t_{n+1}^{n+1} z^n + t_n^{n+1} z^{n+1} + \cdots \\ + t_2^{n+1} z^{2n-1} + t_1^{n+1} z^{2n}. \end{aligned} \tag{3}$$

This, with (2), proves the claim.

(ii) The row-sums σ_n are obviously given by

$$\sigma_n = f_{n-1}(1),\ n = 1, 2, \ldots;$$

hence, by (2), the σ_n satisfy the recursion

$$\sigma_1 = 1,\ \sigma_2 = 3,\ \sigma_{n+1} = 3\sigma_n - \sigma_{n-1};$$

but this is the well-known recursion for the even-indexed Fibonacci numbers F_{2n}; hence, $\sigma_n = F_{2n}$ for $n = 1, 2, \ldots$.

(iii) Claim: The k^{th} differences of the sequence d_{k+1} are eventually all 1.

Proof: Obviously, the numbers $d_{k+1}(n)$ in the sequence d_{k+1} are the $(k+1)^{\text{th}}$ coefficients (counting from the constant coefficient upward) of the polynomials $f_n(z)$, as defined by (1). They can be found by taking the k^{th} derivative of $f_n(z)$:

$$d_{k+1}(n) = \frac{1}{k!} f_n^{(k)}(0). \tag{4}$$

We consider the generating function, see (1):

$$\frac{d^k}{dz^k} G(z, t)\Big|_{z=0} = \sum_{n=0}^{\infty} f_n^{(k)}(0) t^n. \tag{5}$$

To evaluate the left-hand side of (5), we use the partial fraction expansion

$$G(z, t) = \frac{1}{\sqrt{t^2 + 4t(1-t)^2}}\left[\frac{1}{z + \alpha(t)} - \frac{1}{z + \beta(t)}\right],$$

where

$$\alpha(t) := \frac{1}{2(1-t)} + \sqrt{\frac{1}{4(1-t)^2} + \frac{1}{t}},$$

$$\beta(t) := \frac{1}{2(1-t)} - \sqrt{\frac{1}{4(1-t)^2} + \frac{1}{t}},$$

which is easy to verify. Hence,

$$\frac{d^k}{dz^k} G(z, t) = \frac{(-1)^k k!}{\sqrt{t^2 + 4t(1-t)^2}}[(z + \alpha(t))^{-k-1} - (z + \beta(t))^{-k-1}],$$

$$\frac{d^k}{dz^k} G(z, t)\Big|_{z=0} = \frac{(-1)^k k!}{\sqrt{t^2 + 4t(1-t)^2}} \; \frac{(\beta(t))^{k+1} - (\alpha(t))^{k+1}}{(\alpha(t)\beta(t))^{k+1}}$$

$$= \frac{k!}{(1-t)^{k+1}} g_k(t),$$

where

$$g_k(t) = \frac{1}{2^{k+1}\sqrt{t^2 + 4t(1-t)^2}}[(t + \sqrt{t^2 + 4t(1-t)^2})^{k+1} - (t - \sqrt{t^2 + 4t(1-t)^2})^{k+1}]. \tag{6}$$

Hence, with (4) and (5), we have

$$\frac{1}{(1-t)^{k+1}} g_k(t) = \sum_{n=0}^{\infty} d_{k+1}(n) t^n. \tag{7}$$

It is easy to see from (6) that

$$g_k(1) = 1. \tag{8}$$

Using the binomial theorem, we can rewrite (6) as

$$g_k(t) = 2^{-k} \sum_{j=0}^{[k/2]} \binom{k+1}{2j+1} t^{k-j}(t + 4(1-t)^2)^j; \tag{9}$$

this shows that $g_k(t)$ is a polynomial of degree $k + [k/2]$.

The k^{th} difference $\Delta_k(n)$ of the sequence $\{d_{k+1}(n)\}_n$ is

$$\Delta_k(n) := \sum_{j=0}^{k} \binom{k}{j} (-1)^j d_{k+1}(n-j).$$

To evaluate it, we consider the generating function

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_k(n) t^n &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} (-1)^j d_{k+1}(n-j) \right) t^n \\ &= \sum_{j=0}^{k} \binom{k}{j} (-1)^j \sum_{n=0}^{\infty} d_{k+1}(n-j) t^n \\ &= \sum_{j=0}^{k} \binom{k}{j} (-1)^j t^j \sum_{n=0}^{\infty} d_{k+1}(n-j) t^{n-j} \\ &= (1-t)^k g_k(t) \frac{1}{(1-t)^{k+1}} \quad \text{[by (7)]}.\end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty} \Delta_k(n) t^n = \frac{1}{1-t} g_k(t). \tag{10}$$

Finally, if we denote the coefficients of $g_k(t)$ by $a_j^{(k)}$, we get

$$\frac{1}{1-t} g_k(t) = \left(\sum_{n=0}^{\infty} t^n \right) \left(\sum_{j=0}^{k+[k/2]} a_j^{(k)} t^j \right) = \sum_{n=0}^{\infty} t^n \left(\sum_{j=0}^{n} a_j^{(k)} \right),$$

where we set $a_j^{(k)} := 0$ for $j > k + [k/2]$. Hence, by (8), the coefficients in the Taylor series for $g_k(t)/(1-t)$ are all 1 for $n \geq k + [k/2]$. Comparing coefficients on both sides of (10), we get

$$\Delta_k(n) = 1 \text{ for } n \geq k + [k/2],$$

which completes the proof.

(iv) Consider the generating function

$$\frac{1 - zt}{1 - t(1 + z + z^2) + z^2 t^2} = \sum_{n=0}^{\infty} h_n(z) t^n.$$

By multiplying both sides by the denominator of the left-hand side and comparing coefficients, we get the recursion

$$\begin{aligned}&h_0(z) = 1,\ h_1(z) = 1 + z^2, \text{ and} \\ &h_{n+1}(z) = (1 + z + z^2) h_n(z) - z^2 h_{n-1}(z).\end{aligned} \tag{11}$$

Hence, the coefficients of the $h_n(z)$ satisfy the same generating rule as those of $f_n(z)$, and we obtain the triangle

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 0 & 1 & & & \\
 & & 1 & 1 & 1 & 1 & 1 & & \\
 & 1 & 2 & 2 & 3 & 2 & 2 & 1 & \\
1 & 3 & 4 & 6 & 6 & 6 & 4 & 3 & 1 \\
 & & & & \vdots & & & &
\end{array}$$

If τ_n denotes the sum of the elements in the n^{th} row, we have, as in (i),

$$\tau_n = h_{n-1}(1),$$

so, by (11), the τ_n satisfy the recursion

$$\tau_1 = 1,\ \tau_2 = 2,\ \tau_{n+1} = 3\tau_n - \tau_{n-1},$$

and this is the recursion for the odd-indexed Fibonacci numbers F_{2n-1}; hence, $\tau_n = F_{2n-1}$ for $n = 1, 2, \ldots$.

Remark: K. B. Stolarsky considered the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}$$

and its difference analogue

$$2u(x, t + 1) = u(x - 1, t) + u(x, t) + u(x + 1, t) - u(x, t - 1)$$

which, after normalizing, leads to the triangle

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & 1 & 2 & 1 & 2 & 1 & & \\
 & 1 & 3 & 2 & 3 & 2 & 3 & 1 & \\
1 & 4 & 4 & 4 & 5 & 4 & 4 & 4 & 1 \\
 & & & & \vdots & & & &
\end{array}$$

The generating rule for this triangle is very similar to that of T, namely, each element in the n^{th} row is the sum of the three closest elements in the $(n - 1)^{\text{th}}$ row minus *twice* the closest element in the $(n - 2)^{\text{th}}$ row. The generating function in this case is

$$[1 - t(1 + z + z^2) + 2z^2t^2]^{-1},$$

and the sum of the entries in the n^{th} row is $2^n - 1$.

Stolarsky also suggested to study the general case

$$[1 - t(1 + z + z^2) + \lambda z^2t^2]^{-1},$$

where the corresponding triangle is generated as before, with the difference that λ *times* the closest element in the $(n - 2)^{\text{th}}$ row is subtracted. This was carried out in [1], in a slightly more general setting. The sum of the entries of the n^{th} row turns out to be

$$\frac{\lambda^{n/2}}{\sqrt{9 - 4\lambda}}\left\{\left(\frac{3 + \sqrt{9 - 4\lambda}}{2\sqrt{\lambda}}\right)^n - \left(\frac{3 - \sqrt{9 - 4\lambda}}{2\sqrt{\lambda}}\right)^n\right\}.$$

For $\lambda = 1$, the Fibonacci connection becomes apparent again, since

$$(3 \pm \sqrt{5})/2 = ((1 \pm \sqrt{5})/2)^2.$$

Asymptotic formulas for the elements in the columns of the triangle are also given in [1]. For example, for $\lambda < 9/4$, the column elements in the (general) triangle are asymptotically

$$\frac{1}{2\sqrt{\pi(n-1)}}(9-4\lambda)^{-1/4}\left(\frac{3+\sqrt{9-4\lambda}}{2}\right)^n.$$

regardless of which column is considered; n denotes the row, numbered as in the problem. In particular, for $\lambda = 1$, this is

$$\frac{1}{2\sqrt{\pi(n-1)}}\,5^{-1/4}\left(\frac{3+\sqrt{5}}{2}\right)^n \sim \frac{5^{1/4}}{2\sqrt{\pi(n-1)}}F_{2n}.$$

Reference

1. K. Dilcher. "Polynomials Related to Expansions of Certain Rational Functions in Two Variables." *SIAM J. Math. Anal.* (to appear).

Also solved by J.-Z. Lee & J.-S. Lee and the proposer.

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