THE GENERALIZED ZECKENDORF THEOREMS

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We recall the Zeckendorf Theorem and its dual, credited to E. Zeckendorf, which deals with the representation of integers as sums of distinct Fibonacci numbers. These theorems were restated and proved by J. L. Brown, Jr., in [1] and [2]. Throughout this paper, we let N denote the set of positive integers.

Zeckendorf Theorem: If $n \in N$, n may be uniquely expressed in the following form:

$$n = \sum_{k=1}^{r} \theta_k F_{k+1}, \tag{1}$$

where

$$\theta_k \in \{0, 1\},\ \theta_k = 0 \text{ if } k > r, \text{ and } \theta_k + \theta_{k+1} < 2,\ k = 1, 2, \ldots . \tag{2}$$

Dual Zeckendorf Theorem: If $n \in N$, n may be uniquely expressed in the form shown in (1), but with the conditions:

$$\theta_k \in \{0, 1\},\ \theta_k = 0 \text{ if } k > r, \text{ and } \theta_k + \theta_{k+1} > 0,\ k = 1, 2, \ldots, r. \tag{3}$$

[Note: The usual statement of the condition on the θ_k's in (2) is, $\theta_k\theta_{k+1} = 0$, which is equivalent. The condition as stated in (2) is more amenable to the proper generalization.]

Before stating and proving the appropriate generalizations of the above theorems, we introduce some useful definitions.

Given integers b and t with $b \geq 2$, $t \geq 2$, we say that a given integer $n \in N$ is b, *t-upper representable* iff there exists an increasing sequence

$$H = H_k(b, t)_{k=1}^{\infty}$$

of positive integers such that n may be uniquely expressed in the following form:

$$n = \sum_{k=1}^{r} \theta_k(b, t) H_k(b, t), \tag{4}$$

where

$$\theta_k(b, t) \in \{0, 1, \ldots, b-1\}, \quad \theta_k(b, t) = 0 \text{ if } k > r, \tag{5}$$

and

$$\theta_k + \theta_{k+1} + \cdots + \theta_{k+t-1} < (b-1)t,\ k = 1, 2, \ldots . \tag{6}$$

We say that $n \in N$ is b, *t-lower representable* iff the same conditions hold as in (4) and (5), but (6) is replaced by:

$$\theta_k + \theta_{k+1} + \cdots + \theta_{k+t-1} > 0,\ k = 1, 2, \ldots, r. \tag{7}$$

Let $S(H)$ and $T(H)$ denote the sets of b, t-upper representable and b, t-lower representable numbers, respectively. For brevity, we may write the sum in (4) in the form:

$$n = (\theta_r \theta_{r-1} \ldots \theta_2 \theta_1)_H, \tag{8}$$

omitting the arguments "b, t" where no confusion is likely to arise. We may let the notation in (8) represent the b, t-representation of n [an element of $\overline{S}(H)$ or $\overline{T}(H)$] as well as the value of the sum indicated in (4) [an element of $S(H)$ or $T(H)$]. Here, $\overline{S}(H)$ and $\overline{T}(H)$ denote the sets of b, t-upper and -lower representations, respectively, of the form given in (8). Note that condition (6) for b, t-upper representations states that no representation in $\overline{S}(H)$ is to contain t consecutive digits equal to $(b - 1)$; similarly, condition (7) requires that no element of $\overline{T}(H)$ is to contain t consecutive digits equal to zero.

Let $\overline{S}_r(H)$ and $\overline{T}_r(H)$ denote the subsets of $\overline{S}(H)$ and $\overline{T}(H)$, respectively, which contain r digits in the representation (that is, with $\theta_r > 0$, $\theta_k = 0$, if $k > r \geq 1$). Let the corresponding integers represented by $\overline{S}_r(H)$ and $\overline{T}_r(H)$ be arranged in nondecreasing order (as yet, we do not know if any duplication occurs), and call these ordered sets $S_r(H)$ and $T_r(H)$, respectively. Let $U_r(H)$ and $V_r(H)$ denote the sizes of $\overline{S}_r(H)$ and $\overline{T}_r(H)$, respectively, that is,

$$U_r(H) = |\overline{S}_r(H)|, \quad V_r(H) = |\overline{T}_r(H)|. \tag{9}$$

Let $A_r(H)$ and $B_r(H)$ denote the smallest and largest values, respectively, of $S_r(H)$; let $C_r(H)$ and $D_r(H)$ denote the smallest and largest values, respectively of $T_r(H)$. Finally, we observe that:

$$S(H) = \bigcup_{r=1}^{\infty} S_r(H), \quad T(H) = \bigcup_{r=1}^{\infty} T_r(H). \tag{10}$$

We may now express and prove the following theorems.

Theorem 1 (Generalized Zeckendorf): We define the sequence $G = (G_k(b, t))_{k=1}^{\infty}$ as follows:

$$G_k = b^{k-1}, \ k = 1, 2, \ldots, t; \tag{11}$$

$$G_{k+t} = (b - 1)(G_{k+t-1} + G_{k+t-2} + \cdots + G_{k+1} + G_k), \ k = 1, 2, \ldots . \tag{12}$$

Then

$$N = S(G). \tag{13}$$

Moreover, if $N = S(H)$ for some sequence $H = (H_k(b, t))_{k=1}^{\infty}$, then $H = G$.

Theorem 2 (Generalized Dual Zeckendorf): If G is as defined in (11) and (12), then $N = T(G)$. Moreover, if $N = T(H)$ for some sequence $H = (H_k(b, t))_{k=1}^{\infty}$, then $H = G$.

Proof of Theorem 1: We begin by deriving the values of $U_r(H)$. Since

$$\theta_1 \in \{1, 2, \ldots, b - 1\} \text{ if } r = 1,$$

we have

$$U_1(H) = b - 1 = G_2 - G_1.$$

If $r = 2$ (with $t > 2$), then

$$\theta_1 \in \{0, 1, 2, \ldots, b - 1\} \text{ and } \theta_2 \in \{1, 2, \ldots, b - 1\},$$

independently, so

$$U_2(H) = b(b - 1) = G_3 - G_2.$$

Continuing in this fashion, we see that

$$U_r(H) = b^{r-1}(b - 1) = G_{r+1} - G_r, \ r = 1, 2, \ldots, t - 1.$$

Setting $k = 1$ in (12) yields:

$$G_{t+1} = (b - 1)(b^{t-1} + b^{t-2} + \cdots + 1) = b^t - 1.$$

Also, note that $\overline{S}_t(H)$ may be generated by $(b - 1)$ choices for θ_t and b choices for each of $\theta_{t-1}, \theta_{t-2}, \ldots, \theta_1$; however, we must subtract from this composition the (one) choice where all digits are equal to $(b - 1)$. Therefore,

$$U_t(H) = b^{t-1}(b - 1) - 1 = b^t - 1 - b^{t-1} = G_{t+1} - G_t.$$

So far, we have shown:

$$U_r(H) = G_{r+1} - G_r, \quad r = 1, 2, \ldots, t. \tag{14}$$

Next (for brevity, omitting the argument "H"), assuming $m \geq t$, we let $\overline{S}'_m$ and $\overline{S}''_m$ denote the subsets of $\overline{S}_m$ with initial digit in $\{1, 2, \ldots, b - 2\}$ and equal to $(b - 1)$, respectively. Let U'_m and U''_m denote the sizes of $\overline{S}'_m$ and $\overline{S}''_m$, respectively. Also, let

$$W_m = U_1 + U_2 + \cdots + U_m, \quad W'_m = U'_1 + U'_2 + \cdots + U'_m.$$

Now $\overline{S}_m = \overline{S}'_m \cup \overline{S}''_m$; thus, $U_m = U'_m + U''_m$. In what follows, we let x represent any of the digits in $\{1, 2, \ldots, b - 2\}$, $y = (b - 1)$, and 0 the zero digit; also, z represents either x or y. We note that $\overline{S}''_m$ may be formed in any of the following (mutually exclusive and exhaustive) ways:

$$\begin{array}{ccccc}
y\overline{S}'_{m-1} & y0\overline{S}_{m-2} & y00\overline{S}_{m-3} & \cdots\ y00\ldots00\overline{S}_{t-1} & y000\ldots0 \\
yy\overline{S}'_{m-2} & yy0\overline{S}_{m-3} & yy00\overline{S}_{m-4} & \cdots\ yy00\ldots0\overline{S}_{t-2} & yy00\ldots0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\underbrace{yy\ldots y}_{t-1}\overline{S}'_{m-t+1} & \underbrace{yy\ldots y}_{t-1}0\overline{S}_{m-t} & \underbrace{yy\ldots y}_{t-1}00\overline{S}_{m-t-1} & \cdots\ \underbrace{yy\ldots y}_{t-1}00\ldots00\overline{S}_1 & \underbrace{yy\ldots y}_{t-1}00\ldots0
\end{array}$$

Therefore,

$$\begin{aligned}
U''_m = {} & (U'_{m-1} + U'_{m-2} + \cdots + U'_{m-t+1}) + (U_{m-2} + U_{m-3} + \cdots + U_{m-t}) \\
& + (U_{m-3} + U_{m-4} + \cdots + U_{m-t-1}) + \cdots + (U_{t-1} + U_{t-2} + \cdots + U_1) + t - 1 \\
= {} & (W'_{m-1} - W'_{m-t}) + (W_{m-2} - W_{m-t-1}) + (W_{m-3} - W_{m-t-2}) + \cdots + W_{t-1} + t - 1.
\end{aligned}$$

Taking the first difference, we obtain:

$$U''_{m+1} - U''_m = U'_m - U'_{m-t+1} + W_{m-1} - W_{m-t}. \tag{15}$$

Next, we consider the possible ways to generate S'_m, namely, as follows:

$$x\overline{S}_{m-1},\ x0\overline{S}_{m-2},\ x00\overline{S}_{m-3},\ \ldots,\ x00\ldots00\overline{S}_1, \text{ or } x00\ldots00.$$

Since x may be chosen in $b - 2$ ways, we have:

$$U'_m = (b - 2)(U_{m-1} + U_{m-2} + \cdots + U_1 + 1) = (b - 2)(W_{m-1} + 1).$$

Taking first differences in the last expression, we have:

$$U'_{m+1} - U'_m = (b - 2)U_m. \tag{16}$$

Now, adding the expressions in (15) and (16), we obtain:

$$\begin{aligned}
U_{m+1} - U_m &= U'_m - U'_{m-t+1} + W_{m-1} - W_{m-t} + (b - 2)U_m \\
&= (b - 2)(W_{m-1} + 1 - W_{m-t} - 1) + W_{m-1} - W_{m-t} + (b - 2)U_m;
\end{aligned}$$

hence,

$$U_{m+1} = (b - 1)(U_m + W_{m-1} - W_{m-t}) = (b - 1)(W_m - W_{m-t}).$$

Equivalently,

$$U_{m+1} = (b - 1)(U_m + U_{m-1} + \cdots + U_{m-t+1}), \quad m = t, t + 1, t + 2, \ldots . \tag{17}$$

Note that (17) is the same recursion satisfied by the G_m's in (12). Since G_{m+1} and G_m satisfy this recursion, so does $G_{m+1} - G_m$. It follows from (14) and (17) that we have:

$$U_r(H) = G_{r+1} - G_r,\ r = 1, 2, \ldots, \text{ for all } H. \tag{18}$$

Next, we derive expressions for $A_r(H)$ and $B_r(H)$ [recalling that these are the smallest and largest values, respectively, of $S_r(H)$]. For any admissible H, we see that

$$A_r(H) = (1\underbrace{00\ldots0}_{r-1})_H,$$

or, equivalently,

$$A_r(H) = H_r. \tag{19}$$

In particular,

$$A_r(G) = G_r. \tag{20}$$

Also, using the notation introduced earlier, we see that

$$B_r(H) = (\underbrace{yy\ldots y}_{t-1}\ y-1\quad \underbrace{yy\ldots y}_{t-1}\ y-1\ \ldots\ \underbrace{yy\ldots y}_{t-1}\ y-1\quad \underbrace{yy\ldots y}_{v})_H,$$

where $r = ut + v,\ 0 \le v < t$,

and in the above representation there are u blocks of length t of the type:

$$yy\ldots y\ y-1.$$

Therefore,

$$B_r(H) = (b-1)(H_r + H_{r-1} + \cdots + H_1) - (H_{v+1+(u-1)t} + \cdots + H_{v+1}).$$

In particular,

$$\begin{aligned} B_r(G) &= (b-1)\sum_{k=1}^{v} G_k + (b-1)\sum_{j=0}^{u-1}\sum_{k=1}^{t} G_{v+jt+k} - \sum_{j=0}^{u-1} G_{v+1+jt} \\ &= \sum_{k=1}^{v}(b-1)b^{k-1} + \sum_{j=0}^{u-1} G_{v+1+(j+1)t} - \sum_{j=0}^{u-1} G_{v+1+jt} \\ &= b^v - 1 + G_{v+1+ut} - G_{v+1} = b^v - 1 + G_{r+1} - b^v, \end{aligned}$$

or

$$B_r(G) = G_{r+1} - 1. \tag{21}$$

By definition of $A_r(G)$ and $B_r(G)$, we see from (21) that the $S_r(G)$ are disjoint. Moreover, from (20), (21), and (18), we have:

$$B_r(G) - A_r(G) = G_{r+1} - G_r - 1 = U_r(G) - 1. \tag{22}$$

Thus, the difference between the largest and smallest elements of $S_r(G)$ is one less than the number of elements in $\overline{S}_r(G)$. If we can prove that $N \subset S(G)$ (i.e., that all positive integers have a b, t-upper representation, with G the underlying sequence), this in turn will imply that $N = S(G)$. We will need a lemma.

Lemma: $(b-1)G_m < G_{m+1} \le bG_m,\ m = 1, 2, \ldots$.

Proof: The left inequality is clearly true, from (11) and (12). If $1 \le m \le t$, $G_m = b^{m-1}$, so $G_{m+1} = bG_m$ in the range $1 \le m < t$. Also $G_{t+1} = b^t - 1 < bG_t$. Replacing $k + t$ by $m + 1$ and m, respectively, in (12), and subtracting the results, we obtain:

$$G_{m+1} - G_m = (b - 1)(G_m - G_{m-t})$$

or

$$bG_m - G_{m+1} = (b - 1)G_{m-t}, \text{ if } m > t.$$

Therefore, if $m > t$, $bG_m > G_{m+1}$, which yields the right inequality in the statement of the lemma.

Let J_r denote the set $\{1, 2, \ldots, G_r - 1\}$, $r = 2, 3, \ldots$. Assuming $2 \le r \le t$, $G_r = b^{r-1}$, so if $n \in J_r$, n may be uniquely represented as a b-adic number with digits in $\{0, 1, \ldots, b - 1\}$; this representation is also a b, t-upper representation, as well as a b, t-lower representation. Hence,

$$J_r \subset S(G), \quad J_r \subset T(G), \quad \text{if } 2 \le r \le t. \tag{23}$$

Note that $J_1 = \emptyset$, $J_2 = \{1, 2, \ldots, b - 1\}$.

Suppose next that $r \ge t$, and assume $J_r \subset S(G)$; this inductive hypothesis is seen to be true for $r = t$. Given an integer n' with $G_r \le n' < G_{r+1}$, then

$$pG_r \le n' < (p + 1)G_r, \text{ where } 1 \le p \le b - 1.$$

Then $0 \le n' - pG_r < G_r$, so $(n' - pG_r) \in J_r$. Hence, by (23),

$$(n' - pG_r) \in S(G),$$

which implies that

$$n' - pG_r = (\theta_{r-1}\theta_{r-2} \ldots \theta_2\theta_1)_G,$$

which is an element of $T_{r-1}(G)$ (note that $\theta_r = 0$, otherwise $n' - pG_r \ge G_r$, a contradiction). Therefore,

$$n' = (p\theta_{r-1}\theta_{r-2} \ldots \theta_1)_G.$$

A priori, we could have

$$p = \theta_{r-1} = \theta_{r-2} = \cdots = \theta_{r-t+1} = b - 1;$$

if so,

$$n' \ge (b - 1)(G_r + G_{r-1} + \cdots + G_{r-t+1}) = G_{r+1},$$

which would be a contradiction. Hence, $n' \in S(G)$. Therefore, if $r \ge t$ and $J_r \subset S(G)$, we must have the set

$$\{G_r, G_r + 1, G_r + 2, \ldots, bG_r - 1\} \subset S(G).$$

However, by the Lemma, $G_{r+1} \le bG_r$. Therefore, $J_r \subset S(G)$ implies $J_{r+1} \subset S(G)$. Due to (23), it follows by induction that

$$\bigcup_{r=2}^{\infty} J_r \subset S(G).$$

But G is an increasing sequence, so

$$\bigcup_{r=2}^{\infty} J_r = N.$$

Thus, $N \subset S(G)$. By our previous comments, it follows that $N = S(G)$; in other words, there is a 1-to-1 correspondence between N and $S(G)$.

The final part of Theorem 1 states that G is the only sequence generating b, t-upper representations. To prove this, we will assume $N = S(H)$ for some sequence $H = (H_k(b, t))_{k=1}^{\infty}$. Since H must be increasing, and since 1 must have a (unique) representation, it is apparent that $H_1 = 1$. Then, by (18) and (19),

$$U_r(H) = G_{r+1} - G_r \quad \text{and} \quad A_r(H) = H_r.$$

Also, since the $S_r(H)$ must be disjoint, and since all representations must be unique, we must have

$$B_r(H) = A_{r+1}(H) - 1;$$

therefore, by (19), $B_r(H) = H_{r+1} - 1$. Also, however, we see that

$$B_r(H) = U_r(H) + U_{r-1}(H) + \cdots + U_1(H),$$

so

$$B_r(H) = \sum_{k=1}^{r} (G_{k+1} - G_k) = G_{r+1} - G_1 = G_{r+1} - 1.$$

Therefore, $B_r(H) = H_{r+1} - 1 = G_{r+1} - 1$, so $H_{r+1} = G_{r+1}$ for all $r \geq 1$. It follows that $H = G$, which completes the proof of Theorem 1.

Proof of Theorem 2: The proof follows that of Theorem 1. We begin by deriving the values of $V_r(H)$. The initial values of $V_r(H)$ are derived by reasoning identical to that used in the derivation of the initial values of $U_r(H)$, with the exception of $V_t(H)$. Thus,

$$V_r(H) = (b - 1)b^{r-1},\ r = 1, 2, \ldots, t - 1,$$

i.e., in this range, $V_r(H) = (b - 1)G_r$. For $\overline{T}_t(H)$, we must avoid t consecutive zero digits; this will automatically be satisfied if $\theta_t > 0$. Hence,

$$V_t(H) = (b - 1)b^{t-1} = (b - 1)G_t.$$

Thus,

$$V_r(H) = (b - 1)G_r,\ r = 1, 2, \ldots, t. \tag{24}$$

Next, we observe that if $m \geq t$, $\overline{T}_{m+1}(H)$ may be formed in the following mutually exclusive and exhaustive ways (using the same notation as before):

$$z\overline{T}_m,\ z0\overline{T}_{m-1},\ z00\overline{T}_{m-2},\ \ldots,\ z\underbrace{00\ldots0}_{t-1}\overline{T}_{m-t+1}.$$

Since z may be chosen in $(b - 1)$ ways, we have:

$$V_{m+1} = (b - 1)(V_m + V_{m-1} + \cdots + V_{m-t+1}), \tag{25}$$
$$m = t,\ t + 1,\ t + 2,\ \ldots .$$

Note that (25) is the same recursion as satisfied by the G_m's (and the U_m's). We conclude from (24) that

$$V_r(H) = (b - 1)G_r,\ r = 1, 2, \ldots, \text{ for all } H. \tag{26}$$

Next, we derive expressions for $C_r(H)$ and $D_r(H)$, the smallest and largest values, respectively, of $T_r(H)$. We see that, for any admissible H,

$$C_r(H) = (1\underbrace{00\ldots0}_{t-1}\ \ 1\underbrace{00\ldots0}_{t-1}\ \ldots\ 1\underbrace{00\ldots0}_{t-1}\ \ 1\underbrace{00\ldots0}_{v-1})_H,$$
$$\text{where } r = ut + v,\ 1 \leq v \leq t,$$

and the representation above contains u blocks of t digits, of the type

$$1\underbrace{00\ldots0}_{t-1}.$$

Hence,

$$C_r(H) = \sum_{j=0}^{u} H_{v+jt}. \tag{27}$$

Also, it is clear that $D_r(H) = (\underbrace{yy\ldots y}_{r})_H$, or

$$D_r(H) = (b - 1)\sum_{k=1}^{r} H_k. \tag{28}$$

In particular, $D_r(G) = (b - 1)(G_1 + G_2 + \cdots + G_r)$. If $1 \le v \le t - 1$, then

$$\begin{aligned} D_r(G) &= (b-1)\sum_{k=1}^{v} G_k + (b-1)\sum_{j=0}^{u-1}\sum_{k=1}^{t} G_{v+k+jt} \\ &= (b-1)\sum_{k=1}^{v} b^{k-1} + \sum_{j=0}^{u-1} G_{v+1+(j+1)t} \\ &= b^v - 1 + \sum_{j=1}^{u} G_{v+1+jt} = b^v - 1 + \sum_{j=0}^{u} G_{v+1+jt} - G_{v+1} \\ &= b^v - 1 + C_{r+1}(G) - b^v, \end{aligned}$$

or

$$D_r(G) = C_{r+1}(G) - 1, \text{ where } r = ut + v,\ v = 1, 2, \ldots, t - 1. \tag{29}$$

Also, if $v = t$, then $r = (u + 1)t$, so

$$D_r(G) = (b-1)\sum_{k=1}^{(u+1)t} G_k = \sum_{j=1}^{u+1} G_{1+jt};$$

note that in this case

$$\begin{aligned} C_{r+1}(G) &= (\overbrace{1\underbrace{00\ldots0}_{t-1}\ \ 1\underbrace{00\ldots0}_{t-1}\ \ldots\ 1\underbrace{00\ldots0}_{t-1}}^{(u+1)\text{ blocks of } t \text{ digits}}\ 1)_G = \sum_{j=0}^{u+1} G_{1+jt} \\ &= D_r(G) + (G_1 = 1), \end{aligned}$$

which shows that (29) holds also for $v = t$. We may therefore conclude:

$$D_r(G) = C_{r+1}(G) - 1,\ r = 1, 2, \ldots . \tag{30}$$

Note, from (28), that

$$D_r(H) - D_{r-1}(H) = (b - 1)H_r,$$

so

$$D_r(G) - D_{r-1}(G) = (b - 1)G_r = V_r(G).$$

Using (30):

$$D_r(G) - C_r(G) = V_r(G) - 1. \tag{31}$$

We see from (30) that the $T_r(G)$'s are disjoint, by definition of the $C_r(G)$ and $D_r(G)$. Thus, as before, if we can establish that $N \subset T(G)$, (30) and (31) would imply that $N = T(G)$.

Recall that $J_r \subset T(G)$ for $2 \le r \le t$. Suppose next that $r \ge t$, and assume $J_r \subset T(G)$. Given an integer n'' with $G_r \le n'' < G_{r+1}$, it must satisfy

$$pG_r \le n'' < (p + 1)G_r, \text{ where } 1 \le p \le b - 1;$$

then $0 \le n'' - pG_r < G_r$, so $(n'' - pG_r) \in T(G)$, by the inductive hypothesis. Now

$$n'' - pG_r = (\theta_{r-1}\theta_{r-2} \ldots \theta_1)_G,$$

which is an element of $T_{r-1}(G)$ [for, if $\theta_r > 0$, then $(n'' - pG_r) \ge G_r$, a contradiction). Thus,

$$n'' = (p\theta_{r-1}\theta_{r-2} \ldots \theta_1)_G,$$

so $n'' \in T(G)$. Hence, if $r \ge t$ and $J_r \subset T(G)$, we have that

$$\{G_r,\ G_r + 1,\ \ldots,\ bG_r - 1\} \text{ is a subset of } T(G).$$

Since $G_{r+1} \le bG_r$, by the Lemma, $J_r \subset T(G)$ implies $J_{r+1} \subset T(G)$. So, as before, $N \subset T(G)$. By our previous remarks, $N = T(G)$.

To prove that G is the only sequence allowing b, t-lower representations, we suppose that $N = T(H)$ for some sequence H. Then

$$V_r(H) = (b - 1)G_r, \text{ from (26).}$$

Since $N = T(G) = T(H)$, it follows that

$$D_r(H) = C_{r+1}(H) - 1.$$

Also,

$$D_r(H) - D_{r-1}(H) = (b - 1)H_r, \text{ from (28).}$$

But

$$D_r(H) = V_1(H) + V_2(H) + \cdots + V_r(H),$$

so

$$D_r(H) - D_{r-1}(H) = V_r(H) = (b - 1)G_r.$$

From this, it follows that $H_r = G_r$ for all $r \ge 1$, so $H = G$. Q.E.D.

We now illustrate these two theorems with two examples. For $b = t = 2$, we have the "ordinary" Zeckendorf Theorem and its dual, and the appropriate sequence G is the sequence of distinct Fibonacci numbers:

$$\{1, 2, 3, 5, 8, \ldots\} = (F_{k+1})_{k=1}^{\infty}.$$

For $b = 3$, $t = 2$,

$$G = \{1, 3, 8, 22, 60, \ldots\}$$

and we have the following representations:

n	$\overline{S}(G(3, 2))$	$\overline{T}(G(3, 2))$	n	$\overline{S}(G(3, 2))$	$\overline{T}(G(3, 2))$
1	1	1	25	1010	1010
2	2	2	26	1011	1011
3	10	10	27	1012	1012
4	11	11	28	1020	1020
5	12	12	29	1021	1021
6	20	20	30	1100	1022
7	21	21	31	1101	1101
8	100	22	32	1102	1102
9	101	101	33	1110	1110
10	102	102	34	1111	1111
11	110	110	35	1112	1112
12	111	111	36	1120	1120
13	112	112	37	1121	1121
14	120	120	38	1200	1122
15	121	121	39	1201	1201
16	200	122	40	1202	1202
17	201	201	41	1210	1210
18	202	202	42	1211	1211
19	210	210	43	1212	1212
20	211	211	44	2000	1220
21	212	212	45	2001	1221
22	1000	220	46	2002	1222
23	1001	221	47	2010	2010
24	1002	222	48	2011	2011 etc.

For $b = 2$, $t = 3$,

$$G = \{1, 2, 4, 7, 13, 24, 44, \ldots\},$$

which is the sequence of distinct Tribonacci numbers, and we have the following representations:

n	$\overline{S}(G(2, 3))$	$\overline{T}(G(2, 3))$	n	$\overline{S}(G(2, 3))$	$\overline{T}(G(2, 3))$	
1	1	1	26	100010	11110	
2	10	10	27	100011	11111	
3	11	11	28	100100	100100	
4	100	100	29	100101	100101	
5	101	101	30	100110	100110	
6	110	110	31	101000	100111	
7	1000	111	32	101001	101001	
8	1001	1001	33	101010	101010	
9	1010	1010	34	101011	101011	
10	1011	1011	35	101100	101100	
11	1100	1100	36	101101	101101	
12	1101	1101	37	110000	101110	
13	10000	1110	38	110001	101111	
14	10001	1111	39	110010	110010	
15	10010	10010	40	110011	110011	
16	10011	10011	41	110100	110100	
17	10100	10100	42	110101	110101	
18	10101	10101	43	110110	110110	
19	10110	10110	44	1000000	110111	
20	11000	10111	45	1000001	111001	
21	11001	11001	46	1000010	111010	
22	11010	11010	47	1000011	111011	
23	11011	11011	48	1000100	111100	
24	100000	11100	49	1000101	111101	
25	100001	11101	50	1000110	111110	etc.

It is of interest to indicate a generating function for the $G_n(b, t)$'s, namely:

$$F(z; b, t) = \frac{z + z^2 + \cdots + z^t}{1 - (b - 1)(z + z^2 + \cdots + z^t)} = \sum_{n=1}^{\infty} G_n(b, t) z^n. \tag{32}$$

This may be verified by multiplying each side of the last equation by the denominator of the fraction, then applying the relations in (11) and (12) defining $G_n(b, t)$. By multinomial expansion, we may derive the following explicit expression for $G_n(b, t)$ from (32):

$$G_n(b, t) = \sum_{m=1}^{n} (b - 1)^{m-1} \sum_{S} \binom{x_1 + \cdots + x_t}{x_1, \ldots, x_t}, \tag{33}$$

where S is the set of t-ples of nonnegative integers $x_1, x_2, \ldots, x_t$ satisfying

$$x_1 + x_2 + \cdots + x_t = m, \quad x_1 + 2x_2 + \cdots + tx_t = n.$$

We may also show the following result, expressed as a divided difference:

$$G_n(b, t) = (b - 1)^{-1} \Delta^{t-1} z^{n+t-1}(z_1, z_2, \ldots, z_t), \tag{34}$$

where $z_1, z_2, \ldots, z_t$ are the (distinct) roots of the equation:

$$p(z) = p(z;\, b,\, t) = z^t - (b - 1)(z^{t-1} + z^{t-2} + \cdots + 1) = 0. \tag{35}$$

This may be simplified to the following sum:

$$G_n(b,\, t) = (b - 1)^{-1} \sum_{k=1}^{t} z_k^{n+t-1}/p'(z_k). \tag{36}$$

An alternative expression, in terms of a contour integral, is given by:

$$G_n(b,\, t) = (b - 1)^{-1} \frac{1}{2i\pi} \oint_C \frac{z^{n+t-1}}{p(z)}\, dz, \tag{37}$$

where C is any simple closed contour in the complex plane, with positive direction and surrounding $z_1,\, z_2,\, \ldots,\, z_t$ within its interior.

Other expressions may be derived which can be shown to be equivalent, namely:

$$G_n(b,\, t) = \sum_{m=1}^{n} \frac{(b - 1)^{m-1}}{(n - m)!} \cdot \frac{d^{n-m}}{dz^{n-m}}(1 + z + z^2 + \cdots + z^{t-1})^m \bigg|_{z=0}, \tag{38}$$

and

$$G_n(b,\, t) = \sum_{m=1}^{n} (b - 1)^{m-1} \sum_{k=0}^{[(n-m)/t]} (-1)^k \binom{m}{k} \binom{n - 1 - kt}{m - 1}. \tag{39}$$

Undoubtedly, further analysis of such relations should lead to additional interesting results.

References

1. J. L. Brown, Jr. "Zeckendorf's Theorem and Some Applications." *Fibonacci Quarterly 2.3* (1964):163-168.
2. J. L. Brown, Jr. "A New Characterization of the Fibonacci Numbers." *Fibonacci Quarterly 3.1* (1965):1-8.

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