ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-646 *Proposed by A. P. Hillman in memory of Gloria C. Padilla*

We know that $F_{2n} = F_nL_n = F_n(F_{n-1} + F_{n+1})$. Find m as a function of n so as to have the analogous formula $T_m = T_n(T_{n-1} + T_{n+1})$, where T_n is the triangular number $n(n + 1)/2$.

B-647 *Proposed by L. Kuipers, Serre, Switzerland*

Simplify

$$[L_{2n} + 7(-1)^n][L_{3n+3} - 2(-1)^nL_n] - 3(-1)^nL_{n-2}L_{n+2}^2 - L_{n-2}L_{n-1}L_{n+2}^3.$$

B-648 *Proposed by M. Wachtel, Zurich, Switzerland*

The Pell numbers P_n and Q_n are defined by

$$P_{n+2} = 2P_{n+1} + P_n,\ P_0 = 0,\ P_1 = 1;\ Q_{n+2} = 2Q_{n+1} + Q_n,\ Q_0 = 1 = Q_1.$$

Show that $(P_{4n},\ P_{2n}^2 + 1,\ 3P_{2n}^2 + 1)$ is a primitive Pythagorean triple for n in $\{1, 2, \ldots\}$.

B-649 *Proposed by M. Wachtel, Zurich, Switzerland*

Give a rule for constructing a sequence of primitive Pythagorean triples (a_n, b_n, c_n) whose first few triples are in the table

n	1	2	3	4	5	6	7	8
a_n	24	28	88	224	572	1248	3276	7332
b_n	7	45	105	207	555	1265	3293	7315
c_n	25	53	137	305	797	1777	4645	10357

and which satisfy

$$|a_n - b_n| = 17,$$

$$a_{2n-1} + a_{2n} = 26P_{2n} = b_{2n-1} + b_{2n},$$

and $$c_{2n-1} + c_{2n} = 26Q_{2n}.$$

[P_n and Q_n are the Pell numbers of B-648.]

B-650 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome Italy & David Singmaster, Polytechnic of the South Bank, London, UK*

Let us introduce a pair of 1-month-old rabbits into an enclosure on the first day of a certain month. At the end of one month, rabbits are mature and each pair produces $k - 1$ pairs of offspring. Thus, at the beginning of the second month there is 1 pair of 2-month-old rabbits and $k - 1$ pairs of 0-month-olds. At the beginning of the third month, there is 1 pair of 3-month-olds, $k - 1$ pairs of 1-month-olds, and $k(k - 1)$ pairs of 0-month-olds. Assuming that the rabbits are immortal, what is their average age A_n at the end of the n^{th} month? Specialize to the first few values of k. What happens as $n \to \infty$?

B-651 *Proposed by L. Van Hamme, Vrije Universiteit, Brussels, Belgium*

Let $u_0, u_1, \ldots$ be defined by $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = u_{n+1} - u_n$. Also let p be a prime greater than 3, and for n in $X = \{1, 2, \ldots, p - 1\}$, let n^{-1} denote the v in X with $nv \equiv 1 \pmod p$. Prove that

$$\sum_{n=1}^{p-1} (n^{-1}u_{n+k}) \equiv 0 \pmod p$$

for all nonnegative integers k.

SOLUTIONS

Relationship between Variables

B-622 *Proposed by Philip L. Mana, Albuquerque, NM*

For fixed n, find all m such that $L_nF_m - F_{m+n} = (-1)^n$.

Solution by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

Using the Binet forms for L_n and F_m, after some simple manipulations, it can be shown that

$$S_{n,m} = L_nF_m - F_{m+n} = (-1)^n F_{m-n}.$$

It follows that $S_{n,m} = (-1)^n$ iff $F_{m-n} = 1$, that is $m = n - 1$, $n + 1$, $n + 2$.

Also solved by Paul S. Bruckman, Herta T. Freitag, C. Georghiou, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Amitabha Tripathi, and the proposer.

Multiple of L_n

B-623 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S(n) = \sum_{k=1}^{2n-1} L_{n+k}L_k.$$

Prove that $S(n)$ is an integral multiple of L_n for all positive integers n.

Solution by Sahib Singh, Clarion Univ. of Pennsylvania, Clarion, PA

Using the Binet form, $L_{n+k}L_k = L_{n+2k} + (-1)^k L_n$. Thus,

$$\begin{aligned}\sum_{k=1}^{2n-1} L_{n+k}L_k &= (L_{n+2} + L_{n+4} + \cdots + L_{5n-2}) - L_n \\ &= L_{5n-1} - L_{n+1} - L_n \\ &= L_{5n-1} - L_{n-1} - 2L_n.\end{aligned}$$

Since $L_{5n-1} - L_{n-1} = 5F_{2n}F_{3n-1} = 5L_nF_nF_{3n-1}$, $S(n) \equiv 0 \pmod{L_n}$ is true.

Also solved by Paul S. Bruckman, Piero Filipponi, C. Georghiou, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Lawrence Somer, Amitabha Tripathi, and the proposer.

Multiple of F_n^2 or L_n^2

B-624 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$T_n = \sum_{i=1}^{n} L_{2(n+i)-1}.$$

For every positive integer n, prove that either $F_n | T_n$ or $L_n | T_n$.

Solution by Lawrence Somer, Washington, D.C.

We will prove the stronger result that either $F_n^2 | T_n$ or $L_n^2 | T_n$. By the solution to Problem B-605 on page 374 of the November 1988 issue of *The Fibonacci Quarterly*,

$$T_n = (L_{2n} - 2)(L_{2n} + 1).$$

We will show that either $F_n | (L_{2n} - 2)$ or $L_n | (L_{2n} - 2)$. The result will then follow.

It is well known that

$$L_{2n} = L_n^2 - 2(-1)^n \tag{1}$$

and

$$L_n^2 - 5F_n^2 = 4(-1)^n. \tag{2}$$

First, suppose that n is even. Then, by (1) and (2),

$$L_{2n} - 2 = L_n^2 - 4 = 5F_n^2.$$

Thus, $F_n^2 | (L_n - 2)$ if n is even.

Now, suppose that n is odd. Then, by (1),

$$L_{2n} - 2 = (L_n^2 + 2) - 2 = L_n^2,$$

and $L_n^2 | (L_{2n} - 2)$. Q.E.D.

Also solved by Paul S. Bruckman, Piero Filipponi, C. Georghiou, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Amitabha Tripathi, and the proposer.

Recurrences for F_nP_n and L_nP_n

B-625 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let P_0, P_1, ... be the Pell numbers defined by

$$P_0 = 0,\ P_1 = 1,\ P_n = 2P_{n-1} + P_{n-2} \text{ for } n \ge 2.$$

Let $G_n = F_nP_n$ and $H_n = L_nP_n$. Show that (G_n) and (H_n) satisfy

$$K_{n+4} - 2K_{n+3} - 7K_{n+2} - 2K_{n+1} + K_n = 0.$$

Solution by Amitabha Tripathi, SUNY, Buffalo, NY

Let us consider two second-order linear recurrence relations given by

$$x_{n+2} = ax_{n+1} + bx_n,\quad y_{n+2} = cy_{n+1} + dy_n,\quad n \ge 0,$$

with a, b, c, and d complex numbers with at least one of a, c nonzero. Then the sequence $\{z_n\} = \{x_ny_n\}$, $n \ge 0$, is also a linearly recurrent sequence of order at most four. In fact, for $n \ge 0$, we have

$$\begin{aligned} z_{n+4} &= x_{n+4}y_{n+4} = (ax_{n+3} + bx_{n+2})(cy_{n+3} + dy_{n+2}) \\ &= acz_{n+3} + bdz_{n+2} + ady_{n+2}(ax_{n+2} + bx_{n+1}) + bcx_{n+2}(cy_{n+2} + dy_{n+1}) \\ &= acz_{n+3} + (bd + a^2d + bc^2)z_{n+2} + abdx_{n+1}(cy_{n+1} + dy_n) \\ &\qquad + bcdx_{n+2}\frac{y_{n+2} - dy_n}{c} \\ &= acz_{n+3} + (a^2d + 2bd + bc^2)z_{n+2} + abcdz_{n+1} - bd^2y_n(x_{n+2} - ax_{n+1}) \\ &= acz_{n+3} + (a^2d + 2bd + bc^2)z_{n+2} + abcdz_{n+1} - b^2d^2z_n. \end{aligned}$$

The result now follows with $a = b = d = 1$, $c = 2$ for each of the sequences $\{G_n\}$ and $\{H_n\}$.

Also solved by Paul S. Bruckman, Odoardo Brugia & Piero Filipponi, C. Georghiou, L. Kuipers, Y. H. Harris Kwong, Bob Prielipp, Sahib Singh, and the proposer.

Generating Functions for F_nP_n and L_nP_n

B-626 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let G_n and H_n be as in B-625. Express the generating functions

$$G(z) = \sum_{n=0}^{\infty} G_nz^n \quad \text{and} \quad H(z) = \sum_{n=0}^{\infty} H_nz^n$$

as rational functions of z.

Solution by Amitabha Tripathi, SUNY, Buffalo, NY

It is well known (and follows easily from a Binet-type formula for the nth term of a linearly recurrent sequence) that, if

$$f_{n+k} + a_1 f_{n+k-1} + a_2 f_{n+k-2} + \cdots + a_k f_n = 0,$$

then the denominator of the rational expression for the generating function for the sequence f_n is given by the polynomial $(1 + a_1 z + a_2 z^2 + \cdots + a_k z^k)$. Thus,

$$(1 - 2z - 7z^2 - 2z^3 + z^4)K(z)$$
$$= K_0 + (K_1 - 2K_0)z + (K_2 - 2K_1 - 7K_0)z^2 + (K_3 - 2K_2 - 7K_1 - 2K_0)z^3,$$

where $K_{n+4} - 2K_{n+3} - 7K_{n+2} - 2K_{n+1} + K_n = 0$ $(n \ge 0)$. Hence,

$$G(z) = \frac{z - z^3}{1 - 2z - 7z^2 - 2z^3 + z^4} \quad \text{and} \quad H(z) = \frac{z + 4z^2 + z^3}{1 - 2z - 7z^2 - 2z^3 + z^4}.$$

Also solved by Paul S. Bruckman, Odoardo Brogia & Piero Filipponi, C. Georghiou, L. Kuipers, Y. H. Harris Kwong, Sahib Singh, and the proposer.

Integral Mean of Consecutive Cubes

B-627 *Proposed by Piero Filipponi, Fond U. Bordoni, Rome, Italy*

Let

$$C_{n,k} = (F_n^3 + F_{n+1}^3 + \cdots + F_{n+k-1}^3)/k.$$

Find the smallest k in $\{2, 3, 4, \ldots\}$ such that $C_{n,k}$ is an integer for every n in $\{0, 1, 2, \ldots\}$.

Solution by C. Georghiou, University of Patras, Greece

We find that

$$C_{n,k} = [F_{3n+3k-1} - F_{3n-1} + 6(-1)^{n+k}F_{n+k-2} - 6(-1)^n F_{n-2}]/10k.$$

Those k in $\{2, 3, 4, \ldots, 24\}$ such that $k|C_{0,k}$ are in the set $\{6, 9, 11, 19, 24\}$. The only k in the last set such that $k|C_{1,k}$ is $k = 24$. Therefore, the required smallest k is $k \ge 24$. From

$$C_{n+1,k} = C_{n,k} + (F_{n+k}^3 - F_n^3)/k,$$

we get

$$C_{n+1,24} = C_{n,24} + (F_{n+24}^3 - F_n^3)/24$$
$$= C_{n,24} + (F_{n+24}^2 + F_{n+24}F_n + F_n^2)(F_{n+24} - F_n)/24$$
$$= C_{n,24} + 6L_{n+12}(F_{n+24}^2 + F_{n+24}F_n + F_n^2),$$

from which it follows that the answer to the problem is $k = 24$.

Also solved by Paul S. Bruckman, L. Kuipers, Bob Prielipp, Sahib Singh, Amitabha Tripathi, and the proposer.

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