ON THE SCHNIRELMANN DENSITY OF M-FREE INTEGERS

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It is well known that a positive integer is said to be r-free ($r \geq 2$) if it contains no r^{th} power factor greater than 1. Let Q_r denote the set of all r-free integers. If the integers r and k are such that $2 \leq r < k$, an integer of the form $a^k b$, where a is any natural number and b is r-free is called a (k, r)-integer. The set of all (k, r)-integers is denoted by $Q_{k,r}$. The (k, r)-integers were introduced by Cohen [1] and by Subbarao & Harris [6], independently, under different notations. Observe that (∞, r)-integers are the r-free integers; therefore, the (k, r)-integers can be considered as generalized r-free integers.

The Schnirelmann density for a set, S, of positive integers is denoted by $D(S)$. That is,

$$D(S) = \inf_{n \geq 1} \frac{S(n)}{n},$$

where $S(n)$ is the number of integers in S not exceeding n.

Using computational methods, Rogers [5] proved that $D(Q_2) = 53/88$. Duncan [2] showed, by elementary methods, that

$$D(Q_r) > 1 - \sum_p \frac{1}{p^r}, \tag{1}$$

in which the summation is over all primes p. Later, Feng & Subbarao [3] established

$$D(Q_{k,r}) \geq \alpha_{k,r}, \tag{2}$$

where

$$\alpha_{k,r} = \zeta(k)\left(1 - \sum_p \frac{1}{p^r}\right) - \frac{1}{k}\left(1 - \frac{1}{k}\right)^{k-1}, \tag{3}$$

in which $\zeta(k)$ is the Riemann zeta function.

Rieger [4] introduced M-free integers as follows: Suppose M is a set of positive integers with minimal element $r > 1$. A positive integer $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$, where $p_1, p_2, \ldots, p_t$ are distinct primes, is said to be M-free if $\alpha_i \notin M$ for $i = 1, 2, \ldots, t$. The set of all M-free integers is denoted by Q_M.

If r, k are integers such that $2 \leq r < k$, write

$$A = \{r, r+1, r+2, \ldots\},$$

$$B = \{n: n \geq r, n \equiv j \pmod{k} \text{ for some } j\ (r \leq j \leq k-1)\},$$

$$C = \{r\},$$

$$D = \{r, 2r, 3r, \ldots\}.$$

Then observe that $Q_A = Q_r$; $Q_B = Q_{k,r}$, the set of all (k, r)-integers; $Q_C = S_r$, the set of all semi-r-free integers introduced by Suryanarayana [7]; and $Q_D = U_r$, the set of all unitarily r-free integers given by Cohen [1].

The object of this note is to obtain a lower bound for $D(Q_M)$. This bound improves (2) in the case $M = B$. In fact, we prove the following:

Theorem: $D(Q_M) \geq 1 - 2\sum_p (p-1)\sum_{a\in M} p^{-a-1}$.

Proof: If $Q_M(n)$ is the number of integers in Q_M not exceeding n, then

$$Q_M(n) \geq n - \sum_p \alpha_{M,n}(p), \tag{4}$$

where $\alpha_{M,n}(p)$ is the number of integers $m \leq n$ such that $p^a \| m$ for some $a \in M$. To count $\alpha_{M,n}(p)$, for each fixed $a \in M$, we find the number of integers $m \leq n$ with $p^a | m$ and $p^{a+1} \nmid m$, and the latter number is

$$[n/p^a] - [n/p^{a+1}]$$

so that

$$\alpha_{M,n}(p) = \sum_{a\in M}\left(\left[\frac{n}{p^a}\right] - \left[\frac{n}{p^{a+1}}\right]\right) \leq \sum_{a\in M}\left(1 - \frac{1}{p}\right)\left(\left[\frac{n}{p^a}\right] + 1\right). \tag{5}$$

Now, from (4) and (5), we obtain

$$Q_M(n) \geq n - \sum_p \sum_{a\in M}\left(1 - \frac{1}{p}\right)\left(\left[\frac{n}{p^a}\right] + 1\right) \geq n - 2\sum_p (p-1)\sum_{a\in M} n\cdot p^{-a-1},$$

where the sum on the right side is over primes p with $p^a \leq n$ for some $a \in M$, which gives

$$\frac{Q_M(n)}{n} \geq 1 - 2\sum_p (p-1)\sum_{a\in M} p^{-a-1}.$$

Since this is also true when summed over all primes, the theorem follows.

Corollary: For $k > r \geq 2$, $D(Q_{k,r}) \geq b_{k,r}$, where

$$b_{k,r} = 1 - 2\sum_p \frac{p^{k-r} - 1}{p^k - 1}.$$

Proof: Since

$$\sum_{a\in B} p^{-a-1} = \sum_{m=0}^{\infty}\sum_{j=0}^{k-1}\frac{1}{p^{mk+j+1}} = \frac{p^{k-r} - 1}{(p-1)(p^k - 1)}$$

and $Q_B = Q_{k,r}$, the Corollary follows from the Theorem.

Remark 1: For any $k > r \geq 2$, $a_{k,r} < b_{k,r}$. In fact, since

$$\begin{aligned} b_{k,r} &= 1 - 2\sum_p\left(\frac{1}{p^r} - \frac{1}{p^k}\right)\left(1 - \frac{1}{p^k}\right)^{-1} \\ &= 1 - 2\sum_p \frac{1}{p^r}\left(1 + \frac{1}{p^k} + \frac{1}{p^{2k}} + \cdots\right) + 2\sum_p \frac{1}{p^k}\left(1 - \frac{1}{p^k}\right)^{-1} \\ &= 1 - 2\sum_p \frac{1}{p^r} - 2\sum_p \frac{1}{p^{r+k}}\left(1 - \frac{1}{p^k}\right)^{-1} + 2\sum_p \frac{1}{p^k}\left(1 - \frac{1}{p^k}\right)^{-1} \\ &= \left(1 - \sum_p \frac{1}{p^r}\right) - \sum_p \frac{1}{p^r} + 2\sum_p\left(1 - \frac{1}{p^r}\right)\frac{1}{p^k - 1}. \end{aligned}$$

In view of (3), it suffices to show that

$$2\sum_p\left(1-\frac{1}{p^r}\right)\frac{1}{p^k-1} > \sum_p \frac{1}{p^r}+\left(1-\sum_p\frac{1}{p^r}\right)\left(\sum_{n=2}^{\infty}\frac{1}{n^k}\right)$$

$$=\sum_{n=2}^{\infty}\frac{1}{n^k}+\left(\sum_p\frac{1}{p^r}\right)\left(1-\sum_{n=2}^{\infty}\frac{1}{n^k}\right),$$

and this follows if we prove that

$$\sum_{n=2}^{\infty}\frac{1}{n^k}-2\sum_p\left(1-\frac{1}{p^r}\right)\frac{1}{p^k-1} < \left(\sum_{n=2}^{\infty}\frac{1}{n^k}-1\right)\left(\sum_p\frac{1}{p^r}\right). \tag{6}$$

If $a_n = -1$ or 1, according as $n = 1$ or $n > 1$, then $b_n = n^{k-r}$ or 0, according as n is a prime or not and $c_n = [(n^r - 1)/(n^k - 1)]b_n$, so the inequality in (6) can be written as

$$\sum_{n=2}^{\infty}\frac{a_n}{n^k}-2\sum_{n=2}^{\infty}\frac{c_n}{n^k} < \left(\sum_{n=1}^{\infty}\frac{a_n}{n^k}\right)\left(\sum_{n=1}^{\infty}\frac{b_n}{n^k}\right). \tag{7}$$

But, by the multiplication of Dirichlet series, the right side of (7) is:

$$\sum_{n=1}^{\infty}\frac{d_n}{n^k},\text{ where } d_n = \begin{cases}0 & \text{if } n=1,\\ -p^{k-r} & \text{if } n=p, \text{ a prime},\\ \sum\limits_{\substack{p\mid n\\ p<n}} p^{k-r} & \text{otherwise.}\end{cases}$$

Since $d_n > a_n - 2c_n$ for all n, the inequality (7) holds; hence

$$a_{k,r} < b_{k,r}.$$

Thus, the Corollary improves (2). However, the inequality (1) gives a better lower bound for $D(Q_r)$ than the one obtained from the Theorem.

Remark 2: In the special cases of $Q_C = S_r$ and $Q_D = U_r$, defined earlier, the Theorem gives

$$D(S_r) \geq 1 - 2\sum_p \frac{p-1}{p^{r+1}} \quad\text{and}\quad D(U_r) \geq 1 - 2\sum_p\frac{p-1}{p(p^r-1)}.$$

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