A COMBINATORIAL PROBLEM THAT AROSE IN BIOPHYSICS

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The purpose of this note is to prove the following result that was conjectured by T. L. Hill ([1], [2], p. 148) in the course of his investigations of the "surface" properties of some long multi-stranded polymers.

Theorem: Let s be a positive integer, and for any nonnegative integer m, let $R(m)$ be the number of solutions, in *integers* $(m_1, \ldots, m_s)$ of the system

$$m_1 + \cdots + m_s = 0, \tag{1a}$$

$$|m_1| + \cdots + |m_s| = 2m. \tag{1b}$$

Then,

$$Q(\rho) := \sum_{m=0}^{\infty} R(m)\rho^m = (1-\rho)^{-(s-1)} \sum_{k=0}^{s-1} \binom{s-1}{k}^2 \rho^k.$$

Proof: It is readily seen that $R(m)$ is the coefficient of $\rho^m t^0$ in

$$\left[\sum_{k=-\infty}^{\infty} t^k \rho^{|k|/2}\right]^s = [\rho^{1/2} t^{-1}/(1 - \rho^{1/2} t^{-1}) + 1 + \rho^{1/2} t/(1 - \rho^{1/2} t)]^s \tag{2}$$

$$= (1-\rho)^s (1 - \rho^{1/2} t)^{-s} (1 - \rho^{1/2} t^{-1})^{-s}.$$

Thus, $Q(\rho)$ is the coefficient of t^0 in the right side of (2). Expanding the last two terms in the right side of (2) by Newton's binomial formula, and collecting the coefficient of t^0, we get

$$Q(\rho) = (1-\rho)^s \sum_{k=0}^{\infty} \binom{s+k-1}{s-1}^2 \rho^k. \tag{3}$$

Using Euler's transformation for hypergeometric series (e.g., [3], Th. 21, p. 60), (3) can be expressed as the right-hand side of the Theorem. □

The same method of proof can be applied to treat the more general problem where the 0 at the left side of (1a) is replaced by a general integer i.

References

1. T. L. Hill. "Effect of Fluctuating Surface Structure and Free Energy on the Growth of Linear Tubular Aggregates." *Biophysical J.* *49* (1986):1017-1031.
2. T. L. Hill. *Linear Aggregation Theory in Cell Biology.* New York: Springer-Verlag, 1987.
3. Earl D. Rainville. *Special Functions.* New York: Chelsea, 1971.

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