A RADIX PRODUCT REPRESENTATION FOR REAL NUMBERS

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Introduction

Two classical representations for real numbers in terms of integer "digits" are the series representation of Sylvester (1880) and the product representation of Cantor (1869): If A denotes any real number ($A > 1$ in the product case), then these representations, respectively, take the forms:

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots, \qquad a_i \in \mathbb{N},$$

where $\quad a_1 \geq 2,\ a_{i+1} \geq a_i(a_i - 1) + 1 \quad$ for $i \geq 1$,

and $$A = 2^k \prod_{i=1}^{\infty} \left(1 + \frac{1}{a_i}\right), \qquad a_i \in \mathbb{N},$$

where $\quad k \in \mathbb{N},\ a_1 \geq 2,\ a_{i+1} \geq a^2 \quad$ for $i \geq 1$.

For further details, see, for example, Perron [3].

Far more familiar to us than the above is of course the radix or decimal-type representation for A to the base q, where here and throughout, q denotes an integer greater than or equal to two. One of the advantages of this latter representation over the first two, is that the digits "a_i" all lie in the finite set $\{0, 1, \ldots, q - 1\}$ which allows us to conveniently express our decimal expansion base q in the positional notation

$$A = a_n a_{n-1} \ldots a_1 a_0 \cdot a_{-1} a_{-2} a_{-3} \ldots .$$

It seems therefore a natural question to ask whether we can derive a further product representation for a real number $A > 1$ in the radix form

$$A = \prod_{i=-m}^{\infty} \left(1 + \frac{a_i}{q^i}\right), \text{ where } m \in \mathbb{N},\ a_i \in \{0, 1, \ldots, q - 1\}.$$

The paper is set out as follows. In Section 2, we derive a more general type of radix product representation for real numbers $1 < A < 2$. The main interest of the radix product representation is that, like ordinary decimals (base q), it depends only on digits belonging to the set $\{0, 1, \ldots, q - 1\}$, thus allowing us to express the radix product

$$A = \prod_{i=1}^{\infty} \left(1 + \frac{a_i}{q^i}\right)$$

as $0 * a_1 a_2 a_3 \ldots$ say, just as in the decimal case. Furthermore, as shown in the paper, the rate of convergence of the radix product is basically the same as that of the ordinary decimal expansion. It is true that the Cantor product

$$A = \prod \left(1 + \frac{1}{a_i}\right)$$

converges more rapidly. However, by the same token, the Sylvester series

$$A = \sum \frac{1}{a_i}$$

converges far more rapidly than the ordinary decimal expansion. Furthermore, due to the exponential growth of the integers a_i in Sylvester's and Cantor's representation they are unwieldy to use in practice and each "digit" a_i must, in turn, be represented in the decimal system, a drawback which is absent in the case of the radix product. In Section 3, we introduce an alternative, computationally simpler algorithm which allows the computation of the radix product digits from the leading digits of a certain sequence of ordinary q-decimals. Finally, in Section 4, we investigate the possibility of an analogous radix product representation for real numbers $0 < A < 1$.

Throughout the paper, unless otherwise stated, lower case letters denote nonnegative integers.

2. Radix Products in a Varying Scale

Let $q_1, q_2, \ldots$ be an infinite sequence of natural numbers greater than one. Then, it is well known (see, e.g., Perron [3]) that every real number A has a generalized decimal expansion

$$A = a_0 + \frac{a_1}{q_1} + \frac{a_2}{q_1 q_2} + \frac{a_3}{q_1 q_2 q_3} + \ldots,$$

where $a_0 = [A]$, $0 \le a_i \le q_i - 1$ for $i \ge 1$.

Using the product algorithm below, we derive an analogous generalized product representation: Let $1 < A \equiv A_1 < 2$. Then, recursively define, for $n \ge 1$,

$$a_n = [(A_n - 1)q_1 q_2 \ldots q_n],\ A_n \ne 1,$$

where

$$A_{n+1} = \left(1 + \frac{a_n}{q_1 q_2 \ldots q_n}\right)^{-1} A_n.$$

If $A_n = 1$, then stop the algorithm. This leads to

Proposition 2.1: Let $1 < A < 2$. Then A has a finite or infinite product representation

$$A = \prod_{i=1}^{\infty} \left(1 + \frac{a_i}{q_1 q_2 \ldots q_i}\right),$$

where the "digits" a_i satisfy $0 \le a_i \le q_i - 1$.

Proof: First, a repeated application of the recurrence yields

$$A \equiv A_1 = \left(1 + \frac{a_1}{q_1}\right) A_2 = \left(1 + \frac{a_1}{q_1}\right)\left(1 + \frac{a_2}{q_1 q_2}\right) A_3 = \cdots$$

$$= \left(1 + \frac{a_1}{q_1}\right)\left(1 + \frac{a_2}{q_1 q_2}\right) \cdots \left(1 + \frac{a_n}{q_1 q_2 \ldots q_n}\right) A_{n+1},$$

if $A_n \ne 1$. Since $1 < A_1 < 2$, $0 \le a_1 = [(A_1 - 1)q_1] < q_1$. Suppose now, inductively, that $A_i > 1$ and $0 \le a_i \le q_i - 1$ for $i \le n$. From the definition

$$a_n = [(A_n - 1)q_1 q_2 \ldots q_n],$$

we deduce that

$$1 + \frac{a_n}{q_1 \ldots q_n} \le A_n < 1 + \frac{a_n + 1}{q_1 \ldots q_n}$$

and it follows that

$$1 \le A_{n+1} < \left(1 + \frac{a_n + 1}{q_1 \ldots q_n}\right) \Big/ \left(1 + \frac{a_n}{q_1 \ldots q_n}\right) = 1 + \frac{1}{q_1 \ldots q_n + a_n}$$

$$\le 1 + \frac{1}{q_1 \ldots q_n}.$$

Thus,

$$0 \le a_{n+1} = [(A_{n+1} - 1)q_1 \ldots q_{n+1}] < q_{n+1},$$

as required. Now, either $A_n = 1$ for some n, or

$$1 < A_n < 1 + \frac{1}{q_1 q_2 \cdots q_{n-1}} \leq 1 + \frac{1}{2^{n-1}} \to 1 \text{ as } n \to \infty, \text{ for } n \geq 1.$$

The result follows.

Of particular interest to us is the decimal-type product representation obtained by setting $q_1 = q_2 = q_3 = \cdots = q_n$ in the above. Before discussing this case in some detail, we briefly mention one further special product representation of interest, which arises from Proposition 2.1 by setting $q_n = n + 1$ for $n \geq 1$.

Corollary 2.2: Every real number $1 < A < 2$ has a "factorial" product representation

$$A = \prod_{i=1}^{\infty} \left(1 + \frac{a_i}{(i+1)!}\right),$$

where $0 \leq a_i \leq i$ for $i \geq 1$.

In the sequel, however, we shall confine our attention to the most interesting case of Proposition 2.1, obtained by setting $q_n = q$ for all $n \geq 1$.

Theorem 2.3: Every $A > 1$ has a finite or infinite radix product representation (base q) of the form

$$A = \prod_{n=-m}^{\infty} \left(1 + \frac{a_i}{q^i}\right) := a_n a_{m-1} \cdots a_1 a_2 * a_{-1} a_{-2} \cdots,$$

where $m \in \mathbb{N}$, $a_i \in \{0, 1, \ldots, q-1\}$.

Proof: It follows from Proposition 2.1 that we can represent every $1 < A < 2$ as

$$A = \prod_{i=1}^{\infty} \left(1 + \frac{a_i}{q^i}\right).$$

A simple (nonunique) method of extending this product for $1 < A < 2$ to every $A > 1$ is as follows: First, if $A' < 2q$, then, for a suitable $0 \leq a_0 \leq q - 1$, we can write

$$A' = \left(1 + \frac{a_0}{q^0}\right)A,$$

where $1 < A < 2$. Now apply the algorithm to A. Next, if $A'' > 2q$, then there exists $m \in \mathbb{N}$ such that $1 + q^m < A'' \leq 1 + q^{m+1}$. Thus, we can write

$$A'' = \left(1 + \frac{1}{q^{-m}}\right)A',$$

where $1 < A' \leq (1 + q^{m+1})/(1 + q^m) < q$, and the product expansion for A'' now follows from that of $1 < A' < 2q$.

Remarks 2.4: Even in the case $1 < A < 2$ the radix product representation base q is not necessarily unique. For example, to base 2,

$$1 + \frac{1}{2} = \left(1 + \frac{1}{2^2}\right)\left(1 + \frac{1}{2^3}\right)\left(1 + \frac{1}{2^4}\right)\left(1 + \frac{1}{2^8}\right) \cdots,$$

where the one-term expansion on the left follows from applying the algorithm directly to $A = 1.5$, while the algorithm applied to $A = 1.2 = 1.5/1.25$ yields the expansion on the right.

Unfortunately, as these and other examples show, real numbers can have more than one expansion as a radix product subject only to the condition that the digits lie in $\{0, 1, \ldots, q-1\}$. However, the constructive algorithm at the start of Section 3 produces a unique choice for the digits a_i at each step. For the digits produced by this algorithm, it follows from the proof of Proposition 2.1 that the following inequality holds for each $n \geq 1$:

$$(*)\qquad \left(1+\frac{a_n+1}{q^n}\right) > \prod_{i=n}^{\infty}\left(1+\frac{a_i}{q^i}\right).$$

Conversely, it can be shown that there is only one radix product expansion for a given $1 < A < 2$ for which (*) holds for each $n \geq 1$. Thus, every $1 < A < 2$ has a unique radix product expansion

$$A = \prod_{i=1}^{\infty}\left(1+\frac{a_i}{q^i}\right)$$

s.t. for each $i \geq 1$,

$$\left(1+\frac{a_i+1}{q^i}\right) > \prod_{n=i}^{\infty}\left(1+\frac{a_n}{q^n}\right).$$

Furthermore, since the algorithm chooses the largest possible digit "a_i" at each stage, in general, this radix product expansion will converge faster than any other not satisfying (*), and is thus the canonical expansion for A.

In addition, rational numbers need not have finite representations as q-radix products. As a particular case of Euler's product identity

$$1+\frac{1}{y-1} = \prod_{n=1}^{\infty}\left(1+\frac{1}{y^{2^{n-1}}}\right),\ y \in \mathbb{R},\ |y| > 1,$$

we have, for any $r \in \mathbb{N}$,

$$A = 1+\frac{1}{q^r-1} = \prod_{n=1}^{\infty}\left(1+\frac{1}{q^{2^{n-1}r}}\right).$$

Note also that such A have recurring ordinary q-radix expansions of the form $A = 1.\overline{00\ldots01}$, where the period consists of $r-1$ zeros followed by a one. In general, however, other recurring decimals base q need not have "nice" radix product representations, unlike the case above.

3. An Alternative Radix Product Algorithm

We can reformulate the general product algorithm of Section 2 in the case of a fixed base q, into the following computationally simpler form. It is easy to show that the new algorithm is equivalent to that of Section 2 in the case $q_1 = q_2 = \cdots = q$, provided we replace any real number with recurring decimal expansion

$$A = 1+\frac{a}{q^s}+\frac{q-1}{q^{s+1}}+\frac{q-1}{q^{s+2}}+\cdots,\ 0 < a \leq q-1,\ s \in \mathbb{N},$$

by the finite expression

$$A = 1+\frac{a+1}{q^s}.$$

In this form the algorithm determines only the nontrivial digits ($a_i > 0$) in the radix product representation.

If $1 < A < 2$, let $\hat{A}_1 = A$. Then, if the unique decimal expansion of A (base q) is of the form

$$A = 1+\frac{b_1}{q^{r_1}}+\cdots,\ 1 \leq b_1 \leq q-1,\ r_1 \in \mathrm{N},$$

then we can write

$$\hat{A} = 1+\frac{b_1}{q^{r_1}}A_1',\ \text{where } 1 \leq A_1' < 1+\frac{1}{b_1} \leq 2.$$

If

$$\hat{A}_n = 1+\frac{b_n}{q^{r_n}}A_n'$$

has already been defined with

$$1 < A'_n < 1 + \frac{1}{b_n} \le 2,$$

then define

$$\hat{A}_{n+1} = \left(1 + \frac{b_n}{q^{r_n}}\right)^{-1}\hat{A}_n < \left(1 + \frac{b_n}{q^{r_n}}\right)^{-1}\left(1 + \frac{b_n + 1}{q^{r_n}}\right)$$

$$= 1 + \frac{1}{q^{r_n} + b_n} < 1 + \frac{1}{q^{r_n}}.$$

It follows that we can write

$$\hat{A}_{n+1} = 1 + \frac{b_{n+1}}{q^{r_{n+1}}} A'_{n+1},$$

where $r_{n+1} > r_n$, $1 \le b_{n+1} \le q - 1$ and

$$1 \le A'_{n+1} < 1 + \frac{1}{b_{n+1}} \le 2.$$

If $A'_n = 1$, let $\hat{A}_{n+1} = 1$ and stop the algorithm. Then

$$A = \hat{A}_1 = \left(1 + \frac{b_1}{q^{r_1}}\right)\hat{A}_2 = \cdots = \hat{A}_{n+1} \prod_{i=1}^{n} \left(1 + \frac{b_i}{q^{r_i}}\right).$$

If the procedure does not terminate with some $\hat{A}_{n+1} = 1$, then

$$0 < \hat{A}_{n+1} - 1 < \frac{1}{q^{r_{n+1}-1}} \le \frac{1}{q^{r_n}} \le \cdots \le \frac{1}{q^n} \to 0 \text{ as } n \to \infty.$$

Thus, $\lim\limits_{n\to\infty} \hat{A}_{n+1} = 1$, and hence,

$$A = \prod_{i=1}^{\infty} \left(1 + \frac{b_i}{q^{r_i}}\right).$$

if

$$P_n = \left(1 + \frac{b_1}{q^{r_1}}\right) \cdots \left(1 + \frac{b_n}{q^{r_n}}\right),$$

we also have

$$A = \left(1 + \frac{b_{n+1}}{q^{r_{n+1}}} A'_{n+1}\right)P_n$$

and so

$$0 < A - P_n = \frac{b_{n+1}}{q^{r_{n+1}}} A'_{n+1}P_n < \frac{A}{q^{r_{n+1}-1}} < \frac{2}{q^n}.$$

The above argument can therefore be used to give an alternative proof of Theorem 2.3 and, in addition, if for $A > 1$,

$$P_n = \prod_{n=-m}^{n} \left(1 + \frac{b_i}{q^{r_i}}\right),$$

then the rate of approximation to A by the finite "decimal" P_n is given by

$$0 < A - P_n < \frac{A}{q^{r_{n+1}-1}} < \frac{A}{q^n},\ n \ge 1.$$

In order to appreciate how easily this algorithm can be applied in practice, we illustrate it with a numerical example. For convenience, we choose the base q equal to ten: Let $A = \hat{A}_1 = 1.035124$. Then

$P_1 = 0 * 03$, $\hat{A}_2 = (1.03)^{-1}(1.035124) = 1.004974\ldots$,

$P_2 = 0 * 034$, $\hat{A}_3 = (1.004)^{-1}(1.004974\ldots) = 1.000970\ldots$,

$P_3 = 0 * 0349$, $\hat{A}_4 = \ldots$.

To conclude this section, we make a few comments about radix product "fractions" base q, that is, radix products of the form $0 * a_1 a_2 a_3 \ldots$, $0 \le a_i \le q - 1$. It is clear from the above algorithms that any $1 < A \le 2$ has a representation as a fractional radix product. (To obtain a product expansion for $A = 2$, we can apply the algorithm of Section 3 to $A = 1.999\ldots$). However, fractional product representation also exists for certain real numbers greater than two. If we denote the largest such fraction (base q) by

$$\epsilon_q = 0 * (q - 1)(q - 1)\ldots,$$

then it follows from standard inequalities relating infinite series and products that

$$1 + \sum_{n=1}^{\infty} \frac{q - 1}{q^n} < \epsilon_q < \exp\left(\sum_{n=1}^{\infty} \frac{q - 1}{q^n}\right),$$

which gives $2 < \epsilon_q < e$ for every q. However, the actual value of ϵ_q varies with q. In the table below, we list approximations for ϵ_q for some small values of the base q.

TABLE 1. The Largest Radix Product Fraction Corresponding to Given Bases q

q	ϵ_q
2	2.38423
3	2.26971
4	2.20963
5	2.17207
6	2.14619
7	2.12719
8	2.11263
9	2.10110
10	2.09172

Note that the values of ϵ_q listed correspond to those for the finite products

$$\prod_{n=1}^{k} \left(1 + \frac{q - 1}{q^n}\right)$$

for suitable values of k. If we denote such finite products by $\epsilon_q(k)$, then

$$\epsilon_q - \epsilon_q(k) = \epsilon_q(k)\left(\prod_{i=k+1}^{\infty} \left(1 + \frac{q - 1}{q^i}\right) - 1\right)$$

$$< \epsilon_q \left(\exp\left((q - 1) \prod_{i=k+1}^{\infty} \frac{1}{q^i}\right) - 1\right) < e(e^{q^{-k}} - 1).$$

With this as an upper bound for the error, large enough values of k were chosen for each of the entries $q = 2, 3, \ldots, 10$ to give $\epsilon_q - \epsilon_q(k) < 10^{-5}$. Examination of Table 1 suggests that ϵ_q is a decreasing function of q for $q \ge 2$, a fact that can be verified by considering the derivative with respect to q, of $\log \epsilon_q$. Furthermore, using Theorem 5.7, in Hyslop [1] we see that the uniform convergence of the infinite series

$$\sum_{i=1}^{\infty} \frac{q - 1}{q^i} = 1,$$

for $q \ge 2$, implies the uniform convergence of the product ϵ_q, for $q \ge 2$, and it follows that $\lim_{q \to \infty} \epsilon_q = 2$.

4. Radix Product Expansions for Real Numbers Less than One

One immediate product representation for $0 < A < 1$ follows from the radix product expression for $A^{-1} > 1$. Thus, if we have

$$A^{-1} = \prod_{i=-m}^{\infty} \left(1 + \frac{a_i}{q_i}\right), \quad 0 \le a_i \le q - 1,$$

then

$$A = \prod_{i=-m}^{\infty} \left(1 + \frac{a_i}{q^i}\right)^{-1} = \prod_{i=-m}^{\infty} \left(1 - \frac{a_i}{q^i + a_i}\right).$$

In particular, for $A > 1/2$,

$$(1) \qquad A = \prod_{i=1}^{\infty} \left(1 - \frac{a_i}{q^i + a_i}\right).$$

In this form, however, the product no longer has a denominator depending only on the base. This product does, however, suggest the possibility of representing every $0 < A < 1$ in the form

$$A = \prod_{i=1}^{\infty} \left(1 - \frac{b_i}{q^i}\right), \quad 0 \le b_i \le q_i - 1.$$

Unfortunately, it turns out that it is not possible to represent every $0 < A < 1$ or even $1/2 < A < 1$ in this manner.

To see this, let $\{a_k\}$ be a sequence of real numbers with $a_k \in (0, 1)$ for every k. Then we deduce from Weierstrass's inequality (see Mitrinović [2], p. 210):

$$\prod_{n=r}^{k} (1 - a_n) > 1 - \sum_{n=r}^{k} a_n,$$

by taking limits that

$$\prod_{n=2}^{\infty} (1 - a_n) \ge 1 - \sum_{n=2}^{\infty} a_n.$$

Hence,

$$\prod_{n=1}^{\infty} (1 - a_n) \ge \left(1 - \sum_{n=2}^{\infty} a_n\right)(1 - a_1) > 1 - \sum_{n=1}^{\infty} a_n.$$

Applying this last inequality to $p_1 = \prod_{i=1}^{\infty} \left(1 - \frac{q-1}{q^i}\right)$, we obtain

$$p_1 > 1 - \sum_{i=1}^{\infty} \frac{q-1}{q^i} = 0.$$

Since p_1 is the smallest number that can be represented in the form

$$\prod_{i=1}^{\infty} \left(1 - \frac{a_i}{q^i}\right), \quad 0 \le a_i \le q - 1,$$

it follows that there can be no such product representation for any $0 < A < p_1$. Similarly, the largest real number that can be represented in the form

$$\prod_{i=1}^{\infty} \left(1 - \frac{a_i}{q^i}\right), \quad 0 \le a_i \le q - 1, \ a_1 \ne 0,$$

is $p_2 = 1 - (1/q)$, and the smallest real number that can be represented in the form

$$\prod_{i=1}^{\infty} \left(1 - \frac{a_i}{q^i}\right), \quad 0 \le a_i \le q - 1, \ a_1 = 0,$$

is

$$p_3 = \prod_{i=2}^{\infty} \left(1 - \frac{q-1}{q^i}\right).$$

Since the inequality relating infinite products and series yields $p_3 > p_2$ there can again be no such product representation for any real number $p_3 > A > p_2$. In general, since

$$\prod_{i=m+1}^{\infty}\left(1 - \frac{q-1}{q^i}\right) > 1 - \frac{1}{q^m},$$

there will be an infinite sequence of gaps in any representation system based upon products of this type.

A consideration of Equation (1) suggests that, for $1/2 < A < 1$, we can obtain a product expansion with digits in $\{0, 1, \ldots, q - 1\}$ and denominators independent of "a_i" consisting of terms

$$\left(1 - \frac{a_i}{q^i + q}\right), \quad i \geq 1.$$

To obtain such expansions, we introduce the following algorithm: Let

$$\frac{1}{2} < A = A_1 < 1.$$

Then recursively define, for $n \geq 1$,

$$a_n = [(1 - A_n)(q^n + q)], \quad A_n \neq 1,$$

where

$$A_{n+1} = \left(1 - \frac{a_n}{q^n + q}\right)^{-1} A_n.$$

If $A_n = 1$, then stop the algorithm.

Using this we can show, in a similar manner to Proposition 2.1, that

Proposition 4.1: Every $1/2 < A < 1$ has a "near radix" product representation

$$A = \prod_{n=1}^{\infty}\left(1 - \frac{a_n}{q^n + q}\right)$$

with "digits" a_n in the set $\{0, 1, \ldots, q - 1\}$.

References

1. J. M. Hyslop. *Infinite Series*. 5th ed. University Mathematical Texts, 1965.
2. D. S. Mitrinović. *Analytic Inequalities*. New York: Springer-Verlag, 1970.
3. O. Perron. *Irrationalzahlen*. New York: Chelsea, 1951.

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