FIBONACCI NUMBERS OF THE FORM CX^2, WHERE $1 \le C \le 1000$

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Introduction

Let c and n be natural numbers. Let F_n denote the n^{th} Fibonacci number, that is $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \ge 3$. Consider the equation

(*) $\quad F_n = cx^2$.

In [1], Cohn solved (*) for $c = 1, 2$. In [9], we found all solutions of (*) such that c is prime and either $c \equiv 3 \pmod 4$ or $c < 10{,}000$. Harborth & Kemnitz [4] have asked for solutions of (*) for composite values of c. Clearly, it suffices to consider only squarefree values of c.

If $c \le 1000$, then c has at most three distinct odd prime factors. Therefore $c = kp$ where p is prime and k = 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 38, 39, 42, 51, 55, 65, 66, or 70. In this paper, we solve (*) for each of the above values of c. In the cases k = 2, 13, 26, 34, our results are valid only for $p < 10{,}000$; in the other cases, there are no restrictions on p. These results are listed in Table 1. Combining these new results with those from [1] and [9], we obtain all solutions of (*) such that $1 \le c \le 1000$. We list these solutions in Table 2.

Preliminaries

Let p denote a prime, m a natural number. Let L_n denote the n^{th} Lucas number, that is $L_1 = 1$, $L_2 = 3$, $L_n = L_{n-1} + L_{n-2}$ for $n \ge 3$. Let $o_p(n) = k$ if $p^k \| n$, where $k \ge 0$. Let (a/p) denote the Legendre symbol. Let $z(n) = \min\{m : n | F_m\}$. If p is odd and $2 | z(p)$, let $y(p) = \frac{1}{2}z(p)$.

(1) $\quad F_n = x^2$ iff n = 1, 2, or 12.

(2) $\quad F_n = 2x^2$ iff n = 3 or 6.

(3) $\quad$ If $p \equiv 3 \pmod 4$, then $F_n = px^2$ iff $(n, p, x^2) = (4, 3, 1)$.

(4) $\quad$ If $p \equiv 1 \pmod 4$ and $p < 10{,}000$, then $F_n = px^2$ iff (n, p) = (5, 5), (7, 13), (11, 89), (13, 233), (17,1597), or (25,3001).

(5) $\quad F_n \ne 6x^2$.

(6) $\quad L_n = x^2$ iff n = 1 or 3.

(7) $\quad L_n = 2x^2$ iff $n = 6$.

(8) $\quad L_n = 3x^2$ iff $n = 2$.

(9) $\quad L_n \ne 6x^2$.

(10) $\quad L_n = 7x^2$ iff $n = 4$.

(11) $\quad L_n = 11x^2$ iff $n = 5$.

(12) $\quad L_n = 19x^2$ iff $n = 9$

(13) $\quad L_n = 29x^2$ iff $n = 7$.

(14) $\quad L_{4n} \equiv 7 \pmod 8$ if $3 \nmid n$.

(15) $\quad 5 \nmid L_n$, $13 \nmid L_n$, $17 \nmid L_n$ for all n.

(16) $\quad$ If $m \ge 2$, then $m | F_n$ iff $z(m) | n$.

(17) $\quad F_{2n} = F_n L_n$.

(18) $\quad$ If $m \ge 3$, then $F_m | F_n$ iff $m | n$.

(19) $\quad (F_n, L_n) = \begin{cases} 2 & \text{if } 3 | n, \\ 1 & \text{if } 3 \nmid n. \end{cases}$

(20) If $m \ge 2$, then $L_m | L_n$ iff n/m is odd.

(21) $F_{3n}/F_n = L_n^2 - (-1)^n$.

(22) $L_{3n}/L_n = L_n^2 - 3(-1)^n$.

(23) $F_{5n}/5F_n = 5F_n^4 + 5(-1)^n F_n^2 + 1$.

(24) $5 \nmid F_{3n}/F_n$.

(25) $L_{5n}/L_n = L_n^4 - 5(-1)^n L_n^2 + 5$.

(26) $z(p) | (p - e)$ where $e = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod 5, \\ -1 & \text{if } p \equiv \pm 2 \pmod 5, \\ 0 & \text{if } p = 5. \end{cases}$

(27) $(F_m, F_n) = F_{(m, n)}$.

(28) $(F_n, F_{kn}/F_n) | k$.

(29) $(F_n, F_{5n}/5F_n) = 1$.

(30) $F_{4n+1} + 2 = F_{2n-1}L_{2n+2}$.

(31) $F_{4n-1} + 2 = F_{2n+1}L_{2n-2}$.

(32) $(F_m, L_{m \pm n}) | L_n$.

(33) $x^2 - 5y^2 = -4$ iff $x = L_n$, $y = F_n$ for some odd n.

(34) If p is odd, $p | F_m$, and $p \nmid a$, then $o_p(F_{p^k am}/F_n) = k$.

(35) $2 | F_{3m}/F_m$ iff $3 \nmid m$.

(36) $2 | L_n$ iff $3 | n$.

(37) $3 | L_n$ iff $n \equiv 2 \pmod 4$

(38) $4 | L_n$ iff $n \equiv 3 \pmod 6$.

(39) $(F_n, F_{5n}/F_n) = \begin{cases} 5 & \text{if } 5 | n, \\ 1 & \text{if } 5 \nmid n. \end{cases}$

(40) $L_{2n} = L_n^2 - 2(-1)^n$.

(41) If k is odd, then $(L_n, L_{kn}/L_n) | k$.

(42) $o_2(L_n) = \begin{cases} 2 & \text{if } n \equiv 3 \pmod 6, \\ 1 & \text{if } n \equiv 0 \pmod 6, \\ 0 & \text{otherwise.} \end{cases}$

(43) If p is odd, then $p | L_n$ iff $n = ky(p)$, k odd.

(44) $F_{7m}/F_m = 125F_m^6 + 175(-1)^m F_m^4 + 70F_m^2 + 7(-1)^m$.

(45) $3 | F_n$ iff $4 | n$.

Remarks: (6), (7), (1), and (2) are Theorems 1 through 4 in [1]. (3) and (4) are Corollary 1 and Theorem 3 in [9], respectively. (5) and (9) follow from Lemmas 1 and 2 in [20], respectively. (8) and (10) are established in [2], (11) through (13) in [11]. (32) is Theorem 1 in [7]. (28) is Lemma 16 in [3], while (34) follows from Theorem 2 in [3]. (41) follows from Theorem 4 in [8]. (17), (18), (20), and (27) are I_7, Theorem III, Theorem V, and Theorem II in [5], respectively. (40) follows from I_{15} and I_{18} in [5]. The other identities are elementary or well known.

The Main Results

Lemma 1: $L_{3m}/L_m = x^2$ iff $m = 1$.

Proof: If $L_{3m}/L_m = x^2$, then (22) implies $L_m^2 - 3(-1)^m = x^2$. If m is odd, then $L_m^2 = 1$, so $m = 1$. If m is even, then $L_m^2 = 4$, which is impossible, since m is a natural number. Conversely, $L_3/L_1 = 4 = 2^2$.

Lemma 2: $L_{3m}/L_m \ne 2x^2$.

Proof: Assume the contrary. Then (22) implies $L_m^2 - 3(-1)^m = 2x^2$. If $3 | x$, then $3 | L_m$, so we get $\pm 3 \equiv 0 \pmod 9$, an impossibility. If $3 \nmid x$, then $L_m^2 \equiv 2x^2 \equiv 2 \pmod 3$, an impossibility, since $(2/3) = -1$.

Lemma 3: If p is odd, then $F_{mp} \equiv (5/p)F_m \pmod p$.

Proof: This follows from (91) in [6] and Fermat's theorem, noting that $\Delta = 5$ for the Fibonacci sequence.

Lemma 4: If $p \equiv 3$ or $7 \pmod{20}$, then $F_{mp}/F_m \ne x^2$.

Proof: Let $F_{mp}/F_m = x^2$. If $p | F_m$, then (34) implies $o_p(F_{mp}/F_m) = 1$, an impossibility. If $p \nmid F_m$, then Lemma 3 implies $F_{mp}/F_m \equiv (5/p) \pmod{p}$, so $x^2 \equiv (5/p) \pmod{p}$. If $p \equiv 3$ or $7 \pmod{20}$, then $(5/p) = -1$, so $x^2 \equiv -1 \pmod{p}$ and $p \equiv 3 \pmod{4}$, an impossibility.

Lemma 5: If $F_{3m}/F_m = 2x^2$, then m is odd or $m = 2$.

Proof: We must show that $F_{6j}/F_{2j} = 2x^2$ iff $j = 1$. Now, $F_6/F_2 = 8 = 2(2)^2$. If $F_{6j}/F_{2j} = 2x^2$, then (17), (18), and (20) imply $(F_{3j}/F_j)(L_{3j}/L_j) = 2x^2$. If $3 \nmid j$, then (35) implies $2 | F_{3j}/F_j$, so

$$(F_{3j}/2F_j)(L_{3j}/L_j) = x^2.$$

Let $d = (F_{3j}/2F_j,\ L_{3j}/L_j)$. Now, $d | (F_{3j},\ L_{3j})$, so (19) implies $d | 2$. We have

$$F_{3j}/2F_j = dy^2, \quad L_{3j}/L_j = dz^2.$$

Lemma 2 implies $d \ne 2$. Therefore, $d = 1$, so Lemma 1 implies $j = 1$. If $j = 3k$, then (35) implies $2 \nmid F_{9k}/F_{3k}$. Let $g = (F_{9k}/F_{3k},\ L_{9k}/L_{3k})$. Then, $g | (F_{9k},\ L_{9k})$, so (19) implies $g | 2$. But $2 \nmid F_{9k}/F_{3k}$, so $g = 1$. Therefore,

$$F_{9k}/F_{3k} = y^2, \quad L_{9k}/L_{3k} = z^2,$$

which contradicts Lemma 1.

Lemma 6: $F_{3m}/F_m = 3x^2$ iff $(m,\ x^2) = (4,\ 16)$.

Proof: If $F_{3m}/F_m = 3x^2$, then (16) implies $z(3) | 3m$, so $m = 4k$. Now (21) implies $L_{4k}^2 - 1 = 3x^2$. If $3 | k$, then (36) implies $2 | L_{4k}$, so $(L_{4k} + 1,\ L_{4k} - 1) = 1$, so $L_{4k} \pm 1 = u^2$. Now (40) implies $L_{2k}^2 - 1 = u^2$ or $L_{2k}^2 - 3 = u^2$, so $L_{2k}^2 = 1$ or 4, an impossibility. If $3 \nmid k$, then (36) implies $2 \nmid L_{4k}$, so $(L_{4k} + 1,\ L_{4k} - 1) = 2$. In fact, (14) implies

$$\frac{L_{4k} + 1}{8} * \frac{L_{4k} - 1}{2} = 3y^2.$$

Since the factors on the left are coprime, one of them must be a square. If $\tfrac{1}{2}(L_{4k} - 1) = v^2$, then (40) implies $L_{2k}^2 - 3 = 2v^2$, an impossibility, since $(2/3) = -1$. Therefore,

$$(L_{4k} + 1)/8 = u^2 \quad \text{and} \quad \tfrac{1}{2}(L_{4k} - 1) = 3v^2.$$

Now $L_{4k} \equiv 1 \pmod{6}$ implies $(6,\ k) = 1$, so $L_{2k} \equiv 3 \pmod{4}$. (40) implies $L_{2k}^2 - 1 = 8u^2$, so $(L_{2k} + 1)(L_{2k} - 1) = 8u^2$. Since $2 \nmid L_{4k}$, (40) also implies $2 \nmid L_{2k}$, so $(L_{2k} + 1,\ L_{2k} - 1) = 2$. Thus, we have

$$L_{2k} + 1 = 4a^2, \quad L_{2k} - 1 = 2b^2.$$

Again (40) implies $L_k^2 + 3 = 4a^2$, so that $L_k^2 = 1$, $k = 1$, $m = 4$, $x^2 = 16$. Conversely, $F_{12}/F_4 = 144/3 = 48 = 3(4)^2$.

Lemma 7: $F_{3m}/F_m \ne 6x^2$.

Proof: Assuming the contrary and reasoning as in the proof of Lemma 6, we have $m = 4k$ and $L_{4k}^2 - 1 = 6x^2$. Since L_{4k} is odd, (36) and (14) imply

$$((L_{4k} + 1)/8)(L_{4k} - 1)/2) = 6w^2.$$

Since the factors on the left are coprime, we have

$$(L_{4k} + 1)/8 = 2ay^2, \quad \tfrac{1}{2}(L_{4k} - 1) = bz^2, \quad ab = 3.$$

If $a = 1$, then $L_{4k} = (4y)^2 - 1$, which contradicts Theorem 5 in [6]. If $b = 1$, then (14) implies $z^2 \equiv 3 \pmod{4}$, an impossibility.

Lemma 8: If $p | F_{5m}/F_m$, then $p = 5$ or $p \equiv 1 \pmod{10}$.

Proof: If $p | F_{5m}/F_m$ and $p \ne 5$, then $p | F_{5m}/5F_m$, so (23) implies

$$5F_m^4 + 5(-1)^m F_m^2 + 1 \equiv 0 \pmod{p}.$$

Since the discriminant is 5, we must have $(5/p) = 1$; therefore, (26) implies $z(p) | (p - 1)$. Now (16) implies $p | F_{p-1}$. The hypothesis implies $p | F_{5m}$; hence, $p | (F_{5m}, F_{p-1})$. (27) implies $p | F_{(5m, p-1)}$. (29) implies $p \nmid F_m$, so $p \nmid (F_m, F_{p-1})$. (27) also implies $p \nmid F_{(m, p-1)}$; therefore, $(5m, p - 1) \ne (m, p - 1)$, so $5 | (p - 1)$. Thus, $p \ne 2$, so $p \equiv 1 \pmod{10}$.

Lemma 9: $L_{5m}/L_m \ne x^2$.

Proof: Assume the contrary. Then (25) implies

$$L_m^4 - 5(-1)^m L_m^2 + 5 = x^2.$$

The discriminant is $25 - 4(5 - x^2) = 4x^2 + 5$. Since our equation has integer roots, we must have $4x^2 + 5 = t^2$, so $x^2 = 1$, and $L_m^2 = (-1)^m$ or $4(-1)^m$. But then $2 | m$ and $L_m^2 = 1$ or 4, an impossibility.

Lemma 10: If $F_n = x^2 - 2$ and $n \not\equiv 2 \pmod{4}$, then $(n, x^2) = (3, 4)$ or $(9, 36)$.

Proof:

Case 1. Let $n = 4m + 1$. The hypothesis and (30) imply

$$F_{2m-1}L_{2m+2} = x^2.$$

Let $d = (F_{2m-1}, L_{2m+2})$. (32) implies $d | L_3$, that is, $d | 4$. If $d = 1$ or 4, then F_{2m-1} and L_{2m+2} are squares, which contradicts (6). If $d = 2$, then $F_{2m-1} = 2y^2$ and $L_{2m+2} = 2z^2$. (2) implies $2m - 1 = 3$, so $n = 9$ and $x^2 = 36$.

Case 2. Let $n = 4m - 1$. The hypothesis and (31) imply

$$F_{2m+1}L_{2m-2} = x^2.$$

As in Case 1, we must have $(F_{2m+1}, L_{2m-2}) = 2$, so $F_{2m+1} = 2y^2$, $L_{2m-2} = 2z^2$. (2) implies $2m + 1 = 3$, so $n = 3$ and $x^2 = 4$.

Case 3. Let $n = 4m$. Then $F_n \equiv 0$, 3, or 5 (mod 8). But $x^2 - 2 \equiv 6$, 7, or 2 (mod 8). Therefore, $F_n \ne x^2 - 2$.

Lemma 11: $F_{5m}/5F_m = x^2$ iff $m = x^2 = 1$.

Proof: Let $F_{5m}/5F_m = x^2$. If $m = 2k$, then (17), (18), and (20) imply

$$(F_{5k}/5F_k)(L_{5k}/L_k) = x^2.$$

Let $d = (F_{5k}/5F_k, L_{5k}/L_k)$. Then $d | (F_{5k}, L_{5k})$, so (19) implies $d/2$. But Lemma 8 implies $2 \nmid F_{5m}/5F_m$, so $d = 1$. Therefore, both $F_{5k}/5F_k$ and L_{5k}/L_k are squares, which contradicts Lemma 9. If $2 \nmid m$, then (23) implies

$$5F_m^4 - 5F_m^2 + 1 = x^2.$$

The discriminant is $25 - 20(1 - x^2) = 20x^2 + 5$. Since the preceding equation has integer roots, we must have $20x^2 + 5 = t^2$, but then $5 | t$, so $t^2 = 25w^2$, and $4x^2 + 1 = 5w^2$. Therefore $(4x)^2 - 5(2w)^2 = -4$. Now (33) implies that there exists odd n such that $F_n = 2w$, $L_n = 4x$. Also

$$F_m^2 = (5 \pm 5w)/10 = (1 \pm w)/2.$$

Since $F_m^2 > 0$, we have $F_m^2 = \frac{1}{2}(1 + w)$. Therefore, $F_n = 4F_m^2 - 2$. Since n is odd, Lemma 10 implies $F_m = 1$ or 3. Now m is odd, so $F_m \ne 3$. Therefore, $F_m = 1$, so $m = x^2 = 1$. Conversely, $F_5/5F_1 = 1^2$.

Remark: Let $F_m = F_m^* F_m$, where $(F_m^*, F_d) = 1$ for all $d < m$. F_m^* is called the primitive part of F_m. In particular,

$$F^*_{5p} = F_{5p}/F_5F_p = F_{5p}/5F_p \text{ (if } p \ne 5).$$

Lemma 11 implies $F^*_{5p} \ne x^2$.

Lemma 12: $F_{9m}/F_m \ne px^2$.

Proof: If $F_{9m}/F_m = px^2$, let $d = (F_{3m}/F_m,\ F_{9m}/F_{3m})$. Now $d \mid (F_{3m},\ F_{9m}/F_{3m})$; thus, (28) implies $d \mid 3$. If $d = 1$, then F_{3m}/F_m or F_{9m}/F_{3m} is a square, which contradicts Lemma 4. If $d = 3$, then $F_{3k}/F_k = 3y^2$, where $k = m$ or $3m$. Lemma 6 implies $k = 4 = m$. But $F_{36}/F_4 \ne px^2$. The case $F_{9m}/F_m = x^2$ is similar.

Lemma 13: $F_{7m}/F_m \ne 7x^2$.

Proof: Let m be the least integer such that there exists x such that $F_{7m}/F_m = 7x^2$. Now $7 \mid F_{7m}$, so (16) implies $z(7) \mid 7m$, so $8 \mid m$. Let $m = 2k$. (17), (18), and (20) imply

$$(F_{7k}/F_k)(L_{7k}/L_k) = 7x^2.$$

Let $d = (F_{7k}/F_k,\ L_{7k}/L_k)$. Therefore, $d \mid (F_{7k},\ L_{7k})$, so (19) implies $d \mid 2$. But (44) implies F_{7k}/F_k is odd, so $d = 1$. Therefore, $F_{7k}/F_k = y^2$ or $7y^2$. But the first possibility contradicts Lemma 4, while the second possibility contradicts the minimality of m.

Lemma 14: If p and $y(p)$ are odd, then $L_n \ne 2px^2$.

Proof: If $L_n = 2px^2$, then the hypothesis implies $o_2(L_n)$ is odd, so (42) implies $6/n$. But the hypothesis and (43) imply n is odd, a contradiction.

Lemma 15: If $p \equiv 5$ or $7 \pmod 8$, then $L_n \ne 2px^2$.

Proof: Let $L_n = 2px^2$. Then (36) implies $n = 3m$, so that $L_m(L_{3m}/L_m) = 2px^2$. Let $d = (L_m,\ L_{3m}/L_m)$. (41) implies $d \mid 3$.

<u>Case 1</u>. $d = 1$. (22) implies $3 \nmid L_m$, so (37) implies $m \not\equiv 2 \pmod 4$. We have $L_m = ay^2$, $L_{3m}/L_m = bz^2$, with $ab = 2p$, so $a \mid 2$ or $b \mid 2$. If $a = 1$, then $b = 2p$ and (6) implies $m = 1$ or 3. But $L_3/L_1 = 4 \ne 2pz^2$; $L_9/L_3 = 19 \ne 2pz^2$. If $a = 2$, then (7) implies $m = 6$, an impossibility. If $b = 1$, then $a = 2p$ and Lemma 1 implies $m = 1$, so $L_1 = 1 \ne 2pz^2$. Lemma 2 implies $b \ne 2$.

<u>Case 2</u>. $d = 3$. Then $L_m = 3ay^2$, $L_{3m}/L_m = 3bz^2$, with $a \mid 2$ or $b \mid 2$. If $a = 1$, then $b = 2p$, and (8) implies $m = 2$, but $L_6/L_2 = 6 \ne 6pz^2$. (9) implies $a \ne 2$. (37) implies $m \equiv 2 \pmod 4$, so (22) implies $L_m^2 - 3 = 3bz^2$. Therefore, $3bz^2 \equiv -3 \pmod 9$, so $bz^2 \equiv -1 \pmod 3$; thus, $b \ne 1$. If $b = 2$, then $L_m^2 \equiv 3 \pmod 6$, which implies $m = 12k \pm 2$. Since $a = p$, we have $L^2_{12k\pm1} = 3py^2$. (40) implies $L^2_{6k\pm1} + 2 = 3py^2$. Therefore, $(-2/p) = 1$, which is impossible if $p \equiv 5$ or $7 \pmod 8$.

Lemma 16: Let $F_n = kpx^2$, where $2 \mid z(k)$. Then $2 \mid n$ and $F_{\frac{1}{2}n} = day^2$, $L_{\frac{1}{2}n} = dbz^2$, where

$$d = (F_{\frac{1}{2}n},\ L_{\frac{1}{2}n}) = \begin{cases} 2 & \text{if } 3 \mid n \\ 1 & \text{if } 3 \nmid n \end{cases},\quad ab = kp,\ (a,\ b) = 1, \text{ and } dyz = x.$$

Proof: The hypothesis, (16), and (17) imply $2 \mid n$, $F_{\frac{1}{2}n}L_{\frac{1}{2}n} = kpx^2$. The conclusion now follows from (19).

Theorem 1: $F_n \ne 6px^2$.

Proof: Assume the contrary. Then (16) implies $z(6) \mid n$, so $n = 12m$. (38) and Lemma 16 imply $F_{6m} = 4ay^2$, $L_{6m} = 2bz^2$, $ab = 3p$. If $a = 1$, $b = 3p$; hence, (37) implies m is odd. But (1) implies $m = 2$, an impossibility. (3) implies $a \ne 3$. If $b = 1$, then $a = 3p$, so (45) implies $2 \mid m$, but (7) implies $m = 1$, an impossibility. (9) implies $b \ne 3$.

Theorem 2: $F_n = 3px^2$ iff $(n,\ p,\ x^2) = (8,\ 7,\ 1)$ or $(12,\ 3,\ 16)$.

Proof: Assume $F_n = 3px^2$. (16) implies $z(3)|n$, so $n = 4m$. Lemma 16 implies $F_{2m} = day^2$, $L_{2m} = 2bz^2$, $d = (F_{2m}, L_{2m})$, $ab = 3p$. If $3 \nmid m$, then (19) implies $d = 1$, so either $F_{2m} = y^2$ or $3y^2$, or $L_{2m} = z^2$ or $3z^2$. (1), (3), (6), and (8) imply $2m = 2$ or 4, so $n = 4$ or 8. Now $F_4 = 3 \ne 3px^2$. $F_8 = 21 = 3px^2$ implies $p = 7$, $n = 8$, $x^2 = 1$. If $m = 3k$, then (19) implies $d = 2$, so either $F_{6k} = 2y^2$ or $6y^2$, or $L_{2k} = 2z^2$ or $6z^2$. (2), (5), (7), and (9) imply $6k = 6$, so $n = 12$. Now $F_{12} = 144$, so $p = 3$, $n = 12$, $x^2 = 16$. Conversely, $F_8 = 21$ and $F_{12} = 144$.

Theorem 3: Let $2 < p < 10^4$. Then $F_n = 2px^2$ iff $(n, p, x^2) = (9, 17, 1)$.

Proof: If $F_n = 2px^2$, then (16) implies $z(2)|n$, so $n = 3m$ and $F_m(F_{3m}/F_m) = 2px^2$. Let $d = (F_m, F_{3m}/F_m)$. (28) implies $d|3$. If $d = 1$, then $F_m = ay^2$, $F_{3m}/F_m = bz^2$, $ab = 2p$. If $a = 1$, then $2 F_{3m}/F_m$. Therefore, (1) and (35) imply $m = 1$ or 2, so $n = 3$ or 6. But $F_3 = 2 \ne 2px^2$; $F_6 = 8 \ne 2px^2$. If $a = 2$, then $b = p$ and (2) implies $m = 3$ or 6, so $n = 9$ or 18. Now $F_{18}/F_6 \ne px^2$. $F_9/F_3 = 17$, so, if $n = 9$, then $p = 17$, $x^2 = 1$. Lemma 4 implies $b = 1$. If $b = 2$, then $F_m = py^2$. Since $F_2 = 1 = py^2$, Lemma 5 implies m is odd. Therefore, (3), (4), and the hypothesis imply $m = 5, 7, 11, 13, 17$, or 25. But none of the corresponding values of $F_{3m}/2F_m$ is a square. If $d = 3$, then $F_m = 3ay^2$, $F_{3m}/F_m = 3bz^2$, $ab = 2p$. If $a = 1$, then $b = 2p$. (3) implies $m = 4$, but $F_{12}/F_4 = 48 = 6pz^2$, so $p = 2$, contrary to the hypothesis. (5) implies $a \ne 2$. If $b = 1$, then $a = 2p$, which contradicts Theorem 1. If $b = 2$, then $F_{3m}/F_m = 6z^2$, which contradicts Lemma 7. Conversely, $F_9 = 34$.

Theorem 4: $F_n = 5px^2$ iff $(n, p, x^2) = (10, 11, 1)$.

Proof: If $F_n = 5px^2$, then (16) implies $z(5)|n$, so $n = 5m$, and $F_m(F_{5m}/F_m) = 5px^2$, so $F_m(F_{5m}/5F_m) = px^2$. Now (29) implies either (i) $F_m = y^2$, $F_{5m}/5F_m = pz^2$, or (ii) $F_m = py^2$, $F_{5m}/5F_m = z^2$. If (i) holds, then (1) implies $m = 1$, 2, or 12. We get a contradiction unless $m = 2$, $n = 10$, $p = 11$, $x^2 = 1$. If (ii) holds, then Lemma 11 implies $m = 1$, so $F_1 = 1 = py^2$, an impossibility. Conversely, $F_{10} = 55$.

Theorem 5: $F_n = 7px^2$ iff $(n, p, x^2) = (8, 3, 1)$.

Proof: If $F_n = 7px^2$, then (16) implies $z(7)|n$, so $n = 8m$. If $3 \nmid m$, then Lemma 16 implies $F_{4m} = ay^2$, $L_{4m} = bz^2$, $ab = 7p$. If $a = 1$, then (1) implies $m = 3$, a contradiction. (3) implies $a \ne 7$. (6) implies $b \ne 1$. If $b = 7$, then (10) implies $4m = 4$, so $n = 8$, $p = 3$, $x^2 = 1$. If $m = 3k$, then Lemma 16 implies $F_{12k} = 2ay^2$, $L_{12k} = 2bz^2$, $ab = 7p$. (2) implies $a \ne 1$. Theorem 3 implies $a \ne 7$. (7) implies $b \ne 1$. Lemma 15 implies $b \ne 7$. Conversely, $F_8 = 21$.

Theorem 6: $F_n \ne 15px^2$.

Proof: Assume the contrary. Then (16) implies $z(15)|n$, so $n = 20m$. If $3 \nmid m$, then (15) and Lemma 16 imply $F_{10m} = 5ay^2$, $L_{10m} = bz^2$, $ab = 3p$. Now (4) implies $a \ne 1$. Theorem 2 implies $a \ne 3$. (6) and (8) imply $b \ne 1$ and 3, respectively. If $m = 3k$, then (15) and Lemma 16 imply $F_{30k} = 10ay^2$, $L_{30k} = 2bz^2$, $ab = 3p$ Theorems 3 and 1 imply $a \ne 1$ and 3, respectively. (7) and (9) imply $b \ne 1$ and 3, respectively.

Theorem 7: $F_n = 10px^2$ iff $(n, p, x^2) = (15, 61, 1)$.

Proof: If $F_n = 10px^2$, then (16) implies $z(10)|n$, so $n = 15m$, and $F_{5m}(F_{15m}/F_{5m}) = px^2$. Let $d = (F_{5m}, F_{15m}/F_{5m})$. (28) implies $d|3$. (24) implies $F_{5m} = day^2$, $F_{15m}/F_{5m} = dbz^2$, $ab = 2p$. Suppose $d = 1$. If $a = 1$, then $b = 2p$ and (4) implies $5m = 5$, so $F_{15}/F_5 = 122 = 2pz^2$. Therefore, $p = 61$, $n = 15$, $x^2 = 1$. Theorem 3 implies $a \ne 2$. Lemma 4 implies $b \ne 1$. If $b = 2$, then $a = p$, so Theorem 4 implies $5m = 10$. But $F_{30}/F_{10} \ne 2z^2$. Now suppose that $d = 3$. Then $F_5 = 15ay^2$, $F_{15}/F_5 = 3bz^2$, $ab = 2p$. Theorems 2 and 1 imply, respectively, $a \ne 1$ and 2. Lemmas 6 and 7 imply, respectively, $b \ne 1$ and 2. Conversely, $F_{15} = 610$.

Theorem 8: $F_n = 11px^2$ iff $(n, p, x^2) = (10, 5, 1)$.

Proof: If $F_n = 11px^2$, then (16) implies $z(11)|n$, so $n = 10m$. If $3\nmid m$, then Lemma 16 implies $F_{5m} = ay^2$, $L_{5m} = bz^2$, where $ab = 11p$, so a or $b = 1$ or 11. (1) and (3) imply, respectively, $a \ne 1$ and 11. (6) implies $b \ne 1$. If $b = 11$, then $a = p$. Now (11) implies $5m = 5$, so $p = 5$, $n = 10$, and $x^2 = 1$. If $m = 3k$, then Lemma 16 implies $F_{15k} = 2ay^2$, $L_{15k} = 2bz^2$, with a and b as above. (2) implies $a \ne 1$. Theorem 3 implies $a \ne 11$. (6) implies $b \ne 1$. If $b = 11$, then $L_{15k} = 22z^2$. But since $y(11) = 5$, this contradicts Lemma 14. Conversely, $F_{10} = 55$.

Theorem 9: Let $p < 10^4$. Then $F_n = 13px^2$ iff $(n, p, x^2) = (14, 29, 1)$.

Proof: If $F_n = 13px^2$, then (16) implies $z(13)|n$, so $n = 7m$, and $F_m(F_{7m}/F_m) = 13px^2$. Let $d = (F_m, F_{7m}/F_m)$. (28) implies $d|7$. If $d = 1$, then $F_m = ay^2$, $F_{7m}/F_m = bz^2$, $ab = 13p$, so a or $b = 1$ or 13. If $a = 1$, then (1) implies $m = 1$, 2, or 12. We get a contradiction unless $m = 2$, in which case $n = 14$, $p = 29$, $x^2 = 1$. If $a = 13$, then $b = p$ and (4) implies $m = 7$. But $F_{49}/F_7 \ne pz^2$. Lemma 4 implies $b \ne 1$. If $b = 13$, then $a = p$. Now, the hypothesis and (4) imply $m = 4$, 5, 7, 11, 13, 17, or 25. In each case, $F_{7m}/F_m \ne pz^2$. If $d = 7$, then (16) implies $z(7)|m$, so $m = 8k$, and we have $F_{8k} = 7ay^2$, $F_{56k}/F_{8k} = 7bz^2$, $ab = 13p$. (3) implies $a \ne 1$. Theorem 5 implies $a \ne 13$. Lemma 13 implies $b \ne 1$. If $b = 13$, then $a = p$, so Theorem 5 implies $8k = 8$. But then $F_{56}/91F_8 = z^2$, an impossibility. Conversely, $F_{14} = 377$.

Theorem 10: $F_n = 14px^2$ iff $(n, p, x^2) = (24, 23, 144)$.

Proof: If $F_n = 14px^2$, then (16) implies $z(14)|n$, so $n = 24m$. (38) and Lemma 16 imply $F_{12m} = 4ay^2$, $L_{12m} = 2bz^2$, $ab = 7p$. If $a = 1$, then (1) implies $12m = 12$, from which it follows that $n = 24$, $p = 23$, $x^2 = 144$. (3) implies $a \ne 7$. (7) implies $b \ne 1$. Lemma 15 implies $b \ne 7$. Conversely, $F_{24} = 46368$.

Theorem 11: $F_n = 17px^2$ iff $(n, p, x^2) = (9, 2, 1)$.

Proof: If $F_n = 17px^2$, then (16) implies $z(17)|n$, so $n = 9m$ and $F_m(F_{9m}/F_m) = 17px^2$. Let $d = (F_m, F_{9m}/F_m)$. (28) implies $d|9$. Now $F_m = day^2$, $F_{9m}/F_m = dbz^2$, $ab = 17p$. If $d = 1$ or 9, then Lemma 12 implies $b \ne 1$, 17, p. Therefore, $b = 17p$ and $a = 1$, so (1) implies $m = 1$, 2, or 12. We have a contradiction unless $m = 1$, in which case $F_9/17F_1 = 2 = pz^2$, so $p = 2$, $n = 9$, $x^2 = 1$. If $d = 3$, then $o_3(F_{9m}/F\ )$ is odd, but (34) implies $o_3(F_{9m}/F_m) = 2$. Conversely, $F_9 = 34$.

Theorem 12: $F_n \ne 19px^2$.

Proof: Assume the contrary. Then (16) implies $z(19)|n$, so $n = 18m$. Lemma 16 implies $F_{9m} = 2ay^2$, $L_{9m} = 2bz^2$, $ab = 19p$. (2) implies $a \ne 1$. Theorem 3 implies $a \ne 19$. (7) implies $b \ne 1$. Since $y(19) = 9$, Lemma 14 implies $b \ne 19$.

Theorem 13: $F_n = 21px^2$ iff $(n, p, x^2) = (16, 47, 1)$.

Proof: If $F_n = 21px^2$, then (16) implies $z(21)|n$, so $n = 8m$. (37) implies $3\nmid L_{4m}$. If $3\nmid m$, then Lemma 16 implies $F_{4m} = 3ay^2$, $L_{4m} = bz^2$, with $ab = 7p$. If $a = 1$, then (3) implies $4m = 4$, so $L_4 = 7 = 7pz^2$, an impossibility. If $a = 7$, then Theorem 2 implies $4m = 8$ and $L_8 = 47 = pz^2$, so $p = 47$, $n = 16$, and $x^2 = 1$. (6) implies $b \ne 1$. If $b = 7$, then (10) implies $4m = 4$, so $F_4 = 3 = 3pz^2$, an impossibility. If $m = 3k$, then Lemma 16 implies $F_{12k} = 6ay^2$, $L_{12k} = 2bz^2$, $ab = 7p$. (5) implies $a \ne 1$. Theorem 1 implies $a \ne 7$, $a \ne p$. (7) implies $b \ne 1$. Conversely, $F_{16} = 987$.

Theorem 14: $F_n \ne 22px^2$.

Proof: Assume the contrary. Then (16) implies $z(22)|n$, so $n = 30m$. Lemma 16 implies $F_{15m} = 2ay^2$, $L_{15m} = 2bz^2$, $ab = 22p$, so $a|22$ or $b|22$. Now (2) and (1) imply $a \ne 1$ and 2, respectively. Theorem 3 implies $a \ne 11$. (3) implies

$a \ne 22$. (7) and (6) imply $b \ne 1$ and 2, respectively. Lemma 14 implies $b \ne 11$. (11) implies $b \ne 22$.

Theorem 15: $F_n \ne 23px^2$.

Proof: Assume the contrary. Then (16) implies $z(23)|n$, so $n = 24m$. Lemma 16 implies $F_{12m} = 2ay^2$, $L_{12m} = 2bz^2$, $ab = 23p$. (2) implies $a \ne 1$. Theorem 3 implies $a \ne 23$. (7) implies $b \ne 1$. Lemma 15 implies $b \ne 23$.

Theorem 16: Let $p < 10^4$. Then $F_n = 26px^2$ iff $(n, p, x^2) = (21, 421, 1)$.

Proof: If $F_n = 26px^2$, then (16) implies $z(26)$ n, so $n = 21m$ and $F_{7m}(F_{21m}/F_{7m}) = 26px^2$. Let $d = (F_{7m}, F_{21m}/F_{7m})$. (28) implies $d|3$. (34) implies $13\nmid F_{21m}/F_{7m}$. Therefore, if $d = 1$, we have $F_{7m} = 13ay^2$, $F_{21m}/F_{7m} = bz^2$, $ab = 2p$. If $a = 1$, then (4) implies $7m = 7$, so $F_{21}/2F_7 = 421 = pz^2$. Therefore, $p = 421$, $n = 21$, and $x^2 = 1$. Theorem 3 implies $a \ne 2$. Lemma 4 implies $b \ne 1$. If $b = 2$, then $F_7 = 13py^2$. The hypothesis and Theorem 9 imply $7m = 14$. But $F_{42}/F_{14} \ne 2z^2$. If $d = 3$, then (16) implies $z(3)|7m$, that is, $4|7m$, so $7m = 28k$. We now have $F_{28k} = 39ay^2$, $F_{84k}/F_{28k} = 3bz^2$, with $ab = 2p$. Theorems 2 and 1 imply $a \ne 1$ and 2, respectively. Lemmas 6 and 7 imply $b \ne 1$ and 2, respectively. Conversely, $F_{21} = 10346$.

Theorem 17: $F_n = 29px^2$ iff $(n, p, x^2) = (14, 13, 1)$.

Proof: If $F_n = 29px^2$, then (16) implies $z(29)|n$, so $n = 14m$. If $3\nmid m$, then Lemma 16 implies $F_{7m} = ay^2$, $L_{7m} = bz^2$, $ab = 29p$. (1) implies $a \ne 1$. (4) implies $a \ne 29$. (6) implies $b \ne 1$. If $b = 29$, then $F_{7m} = py^2$. (13) implies $7m = 7$, so $F_7 = 13 = py^2$. Therefore, $p = 13$, $n = 14$, $x^2 = 1$. If $m = 3k$, then Lemma 16 implies $F_{21k} = 2ay^2$, $L_{21k} = 2bz^2$, $ab = 29p$. (2) implies $a \ne 1$. Theorem 3 implies $a \ne 29$. (7) implies $b \ne 1$. Since $y(29) = 7$, Lemma 14 implies $b \ne 29$. Conversely, $F_{14} = 377$.

Theorem 18: $F_n \ne 30px^2$.

Proof: Assume the contrary. Then (16) implies $z(30)|n$, so $n = 60m$. Lemma 16 implies $F_{30m} = 2ty^2$, $L_{30m} = 2bz^2$, $tb = 30p$. But (15) and (42) imply $(b, 10) = 1$, so $F_{30m} = 20ay^2$, $L_{30m} = 2bz^2$, $ab = 3p$. If $a = 1$, then $F_{30m} = 5(2y)^2$, which contradicts (4). Theorem 2 implies $a \ne 3$. (7) implies $b \ne 1$. (9) implies $b \ne 3$.

Theorem 19: $F_n \ne 31px^2$.

Proof: Assume the contrary. Then (16) implies $z(31)|n$, so $n = 30m$. Lemma 16 implies $F_{15m} = 2ay^2$, $L_{15m} = 2bz^2$, $ab = 31p$. (2) implies $a \ne 1$. Theorem 3 implies $a \ne 31$. (7) implies $b \ne 1$. Lemma 15 implies $b \ne 31$.

Theorem 20: $F_n \ne 33px^2$.

Proof: Assume the contrary. Then (16) implies $z(33)|n$, so $n = 20m$. (43) implies $11\nmid L_{10m}$. If $3\nmid m$, then Lemma 16 implies $F_{10m} = 11ay^2$, $L_{10m} = bz^2$, $ab = 3p$. (3) implies $a \ne 1$. Theorem 2 implies $a \ne 3$. (6) and (8) imply $b \ne 1$ and 3, respectively. If $m = 3k$, then Lemma 16 implies $F_{30k} = 22ay^2$, $L_{30k} = 2bz^2$, $ab = 3p$. Theorems 3 and 1 imply $a \ne 1$ and 3, respectively. (7) and (9) imply $b \ne 1$ and 3, respectively.

Theorem 21: If $p < 10^4$, then $F_n = 34px^2$ iff $(n, p, x^2) = (18, 19, 4)$.

Proof: If $F_n = 34px^2$, then (16) implies $z(34)|n$, so $n = 9m$ and $F_{3m}(F_{9m}/F_{3m}) = 34px^2$. Let $d = (F_{3m}, F_{9m}/F_{3m})$. (28) implies $d|3$. (35) implies $2\nmid F_{9m}/F_{3m}$. If $d = 1$, then $F_{3m} = 2ay^2$, $F_{9m}/F_{3m} = bz^2$, $ab = 17p$. If $a = 1$, then $b = 17p$ and (2) implies $3m = 3$ or 6. If $3m = 3$, then $F_9/17F_3 = 1 \ne pz^2$. If $3m = 6$, then $F_{18}/17F_6 = 19 = pz^2$, so $p = 19$; hence, $n = 18$ and $x^2 = 4$. If $a = 17$, then $b = p$, and Theorem 3 implies $3m = 9$. But $F_{27}/F_9 \ne pz^2$. Lemma 4 implies $b \ne 1$.

If $b = 17$, then $F_{3m} = 2py^2$. But the hypothesis and Theorem 3 imply $p = 17$, so $17 | d$, an impossibility. If $d = 3$, then (45) imlies $m = 4k$, so $F_{12k} = 6ay^2$, $F_{36k}/F_{12k} = 3bz^2$, $ab = 17p$. (5) implies $a \ne 1$. Theorem 1 implies $a \ne 17$, p. Lemma 6 implies $b \ne 1$. Conversely, $F_{18} = 2584$.

Theorem 22: $F_n \ne 35px^2$.

Proof: Assume the contrary. Then (16) implies $z(35)|n$, so $n = 40m$. If $3 \nmid m$, then (15) and lemma 16 imply $F_{20m} = 5ay^2$, $L_{20m} = bz^2$, $ab = 7p$. (4) implies $a \ne 1$. Theorem 4 implies $a \ne 7$, $a \ne p$, so $b \ne 7$. (6) implies $b \ne 1$. If $m = 3k$, then (15) and Lemma 16 imply $F_{60k} = 10ay^2$, $L_{60k} = 2bz^2$, $ab = 7p$. Theorem 3 implies $a \ne 1$. Theorem 7 implies $a \ne 7$, $a \ne p$, so $b \ne 7$. (7) implies $b \ne 1$.

We omit the proofs of the two following theorems (23 and 24) because they are similar to proofs of prior theorems.

Theorem 23: $F_n = 38px^2$ iff $(n, p, x^2) = (18, 17, 4)$.

Theorem 24: $F_n \ne 39px^2$.

Theorem 25: $F_n \ne 42px^2$.

Proof: Assume the contrary. Then (16) implies $z(42)|n$, so $n = 24m$. (37) implies $3 \nmid L_{12m}$; (38) implies $4 \nmid L_{12m}$. Therefore, Lemma 16 implies $F_{12m} = 12ay^2$, $L_{12m} = 2bz^2$, $ab = 7p$. (3) implies $a \ne 1$. Theorem 2 implies $a \ne 7$. (7) implies $b \ne 1$. Lemma 15 implies $b \ne 7$.

Theorem 26: $F_n \ne 51px^2$.

Proof: Assume the contrary. Then (16) implies $z(51)\ n$, so $n = 36m$. (15), (37), and Lemma 16 imply $F_{18m} = 102ay^2$, $L_{18m} = 2bz^2$, $ab = p$. Theorem 1 implies $a \ne 1$. (7) implies $b \ne 1$.

Theorem 27: $F_n \ne 55px^2$.

Proof: Assume the contrary. Then (16) implies $z(55)|n$, so $n = 10m$. If $3 \nmid m$, then (15) and Lemma 16 imply $F_{5m} = 5ay^2$, $L_{5m} = bz^2$, $ab = 11p$. If $a = 1$, then Theorem 4 implies $5m = 5$, so $L_5/11 = 1 = pz^2$, an impossibility. If $a = 11$, then Theorem 4 implies $5m = 10$, so $L_{10} = 123 = pz^2$, an impossibility. (6) implies $b \ne 1$. If $b = 11$, then (11) implies $5m = 5$, so $F_{5m}/5 = 1 = py^2$, an impossibility. If $m = 3k$, then (15) and Lemma 16 imply $F_{15k} = 10ay^2$, $L_{15k} = 2bz^2$, $ab = 11p$. Theorem 3 implies $a \ne 1$. Theorem 7 implies $a \ne 11$. (7) implies $b \ne 1$. Lemma 14 implies $b \ne 11$.

Theorem 28: $F_n = 65px^2$ iff $(n, p, x^2) = (35, 141961, 1)$.

Proof: $F_{35} = 65 * 141961 * 1^2$. If $F_n = 65px^2$, then (16) implies $z(65)|n$, so $n = 35m$, and $F_{7m}(F_{35m}/F_{7m}) = 65px^2$. Let $d = (F_{7m}, F_{35m}/F_{7m})$. Now Lemma 8 implies $13 \nmid F_{35m}/F_7$. If $5 \nmid m$, then (39) implies $d = 1$, so $F_{7m} = 13ay^2$, $F_{35m}/F_{7m} = 5bz^2$, $ab = p$. If $a = 1$, then (4) implies $7m = 7$, so $F_{35}/5F_7 = 141961 = pz^2$. Therefore $p = 141961$, $n = 35$, $x^2 = 1$. Lemma 11 implies $b \ne 1$. If $m = 5k$, then (39) implies $d = 5$. (34) implies $5^2 \nmid F_{175k}/F_{35k}$. Thus, $F_{35k} = 325ay^2$, $F_{175k}/F_{35k} = 5bz^2$, $ab = p$. But (4) implies $a \ne 1$. Lemma 11 implies $b \ne 1$.

Theorem 29: $F_n \ne 66px^2$.

Proof: Assume the contrary. Then (16) implies $z(66)|n$, so $n = 60m$. Now (43), (38), and Lemma 16 imply $F_{30m} = 44ay^2$, $L_{30m} = 2bz^2$, $ab = 3p$. (3) implies $a \ne 1$. Theorem 2 implies $a \ne 3$. (7) and (9) imply $b \ne 1$ and 3, respectively.

Theorem 30: $F_n \ne 70px^2$.

Proof: Assume the contrary. Then (16) implies $z(70)|n$, so $n = 120m$. (15), (38), and Lemma 16 imply $F_{60m} = 20ay^2$, $L_{60m} = 2bz^2$, $ab = 7p$. (4) implies $a \ne 1$. Theorem 22 implies $a \ne 7$. (7) implies $b \ne 1$. Lemma 15 implies $b \ne 7$.

We summarize the results of Theorems 1 through 30 in Table 1. For each listed value of k, we list all solutions of (*) with $c = kp$, if any. The cases $k = 2, 23, 26, 34$ are subject to the restriction that $p < 10{,}000$.

TABLE 1

k	(n, p, x^2)	k	(n, p, x^2)	k	(n, p, x^2)	k	(n, p, x^2)	k	(n, p, x^2)
2	(9, 17, 1)	10	(15, 61, 1)	21	(16, 47, 1)	31	***********	42	**************
3	(8, 7, 1)	11	(10, 5, 1)	22	************	33	***********	51	**************
3	(12, 3, 16)	13	(14, 29, 1)	23	************	34	(18, 19, 4)	55	**************
5	(10, 11, 1)	14	(24, 23, 144)	26	(21, 421, 1)	35	***********	65	(35, 141961, 1)
6	***********	15	*************	29	(14, 13, 1)	38	(18, 17, 4)	66	***************
7	(8, 3, 1)	17	(9, 2, 1)	30	************	39	***********	70	***************
		19	*************						

Combining these new results with those of [1] and [9], we obtain Table 2, which lists all solutions of (*) such that $1 \le C \le 1000$.

TABLE 2

c	(n, x^2)	c	(n, x^2)	c	(n, x^2)	c	(n, x^2)
1	(1, 1)	3	(4, 1)	34	(9, 1)	322	(24, 144)
1	(2, 1)	5	(5, 1)	55	(10, 1)	377	(14, 1)
1	(12, 144)	8	(6, 1)	89	(11, 1)	610	(15, 1)
2	(3, 1)	13	(7, 1)	144	(12, 1)	646	(18, 4)
2	(6, 4)	21	(8, 1)	233	(13, 1)	987	(16, 1)

Concluding Remarks

Notice that in every solution we have $x^2 = 1$, 4, or 144. This leads us to conjecture that in any solution of (*) one must have $x^2 = 1$, 4, or 144.

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