ON FERMAT'S EQUATION

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1. Introduction

In 1856 I. A. Grünert ([6], see also [9], p. 226) proved that if n is an integer, $n \geq 2$ and $0 < x < y < z$ are real numbers satisfying the equation

(1.1) $x^n + y^n = z^n$

then

(1.2) $z - y < \frac{x}{n}$.

This result was rediscovered by G. Towes [10], and then by D. Zeitlin [11].

In 1979 L. Meres [7] improved the result of Grünert, replacing (1.2) by

(1.3) $z - y < \frac{x}{a}$, for $a = n + 1 - n^{2-n}$, $n \geq 2$.

In [1], we improved the result of Meres, replacing (1.3) by

(1.4) $z - y < \frac{x}{n+1}$, for $n \geq 4$.

Next, in [2], it has been proved that if k is a positive integer and, for $n > [(2k + 1)C_1]$, $C_1 = (\log 2)/[2(1 - \log 2)]$, Equation (1.1) has a solution in real numbers $0 < x < y < z$, then

(1.5) $z - y < \frac{x}{n+k}$.

Fell, Graz, & Paasche [5] have proved that, if (1.1) has a solution in positive integers $x < y < z$, where $n \geq 2$, then

(1.6) $x^2 > 2y + 1$.

In 1969, M. Perisastri ([8], cf. [9], p. 226) proved that

(1.7) $x^2 > z$.

In [2], it has been proved that

(1.8) $x^2 > 2z + 1$.

A. Choudhry, in [4], improved the inequality (1.8) to the form

(1.9) $x^{1+\frac{1}{n-1}} > z$.

In fact, A. Choudhry proved that

(1.10) $z < C(n) \cdot x^{1+\frac{1}{n-1}}$,

where

(1.11) $C(n) = \frac{2^{\frac{1}{n}}}{n^{\frac{1}{n-1}}}$, for $n \geq 2$.

First we remark that inequality (1.9) in the Theorem of Choudhry follows immediately from (1.1) and the assumption that $0 < x < y < z$. Really, we have

$$x^n = z^n - y^n = (z - y)(z^{n-1} + z^{n-2}y + \cdots + y^{n-1}) > z^{n-1},$$

and (1.9) follows.

In this paper we prove the following theorems.

Theorem 1: If the equation (1.1) has a solution in positive integers $x < y < z$ where $n \geq 2$, then

$$(1.12) \quad z < C_1(n) \cdot x^{1+\frac{1}{n-1}}$$

where

$$(1.13) \quad C_1(n) = \frac{2^{\frac{1}{2n}}}{n^{\frac{1}{n-1}}}.$$

We remark that $C_1(n) < C(n) < 1$.

Next, we have the following theorem.

Theorem 2: If $z - x \leq C$, then (1.1) has only a finite number of solutions in positive integers $x < y < z$ and

$$(1.14) \quad z < C\left(n \cdot 2^{\frac{n-1}{n}} + 1\right).$$

We remark that, from Theorem 1 (see [2]) and the inequality (1.5), we get the following corollary.

Corollary: If k is a positive integer (1.1) has a solution in positive integers $x < y < z$ for $n > [(2k + 1)C_1]$, $C_1 = (\log 2)/[2(1 - \log 2)]$, then

$$x > k + [(2k + 1)C_1].$$

Let $G_2(k)$ be the set of all matrices of the form

$$\begin{pmatrix} r & s \\ ks & r \end{pmatrix},$$

where $k \neq 0$ is a fixed integer and $r, s \neq 0$ are arbitrary integers.

Let R_K denote the ring of all integers of the field $K = Q(\sqrt{k})$. Then, in [3], we proved the following theorem.

Theorem 3: A necessary and sufficient condition for (1.1) to have a solution in elements $A, B, C \in G_2(k)$ is the existence of the numbers $\alpha, \beta, \gamma \in R_K$, where $K = Q(\sqrt{k})$ such that $\alpha^n + \beta^n = \gamma^n$. The proof of Theorem 3 in [3] is based on some properties of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ with } a, b, c, d \in Z.$$

In this paper we give a very simple proof of this theorem.

2. Proof of Theorems

2.1 Proof of Theorem 1

For the proof of Theorem 1, we note that

$$(2.1) \quad z^{n-1} + z^{n-2}y + \cdots + zy^{n-2} + y^{n-1} > n(zy)^{\frac{n-1}{2}}.$$

From (1.1) and $x < y < z$ we have $z^n < 2y^n$; hence,

$$(2.2) \quad y > \left(\frac{1}{2}\right)^{\frac{1}{n}} \cdot z.$$

Since

$$(2.3) \quad x^n = (z - y)(z^{n-1} + z^{n-2}y + \cdots + zy^{n-2} + y^{n-1}),$$

we see, by (2.1), (2.2), and (2.3), that it follows that

(2.4) $x^n > n \cdot z^{n-1}\left(\frac{1}{2}\right)^{\frac{n-1}{2n}}.$

From (2.4), we get

$$z < \frac{2^{\frac{1}{2n}}}{n^{\frac{1}{n-1}}} \cdot x^{1+\frac{1}{n-1}}$$

and the proof is complete.

2.2 Proof of Theorem 2

From (1.1), we have

(2.5) $y^n = (z - x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}).$

Since $x < y < z$, then by (2.5) it follows that

(2.6) $y^n < (z - x)n \cdot z^{n-1}.$

From (2.6) and (2.2), we get

(2.7) $y^n < (z - x)n\left(2^{\frac{1}{n}}y\right)^{n-1} = n \cdot 2^{\frac{n-1}{n}}(z - x)y^{n-1}.$

From (2.7), we get

(2.8) $y < n \cdot 2^{\frac{n-1}{n}}(z - x).$

From (2.8) and our assumption that $z - x \le C$, we have

(2.9) $y < n \cdot 2^{\frac{n-1}{n}}C.$

Since $x < y$, we see by (2.9) that $x < n \cdot 2^{\frac{n-1}{n}}C$. From our assumption, it now follows that

$$z \le x + C < n \cdot 2^{\frac{n-1}{n}}C + C = C\left(1 + n \cdot 2^{\frac{n-1}{n}}\right)$$

and the proof is finished.

2.3 Proof of Theorem 3

First we remark that it suffices to prove that the set $G_2(k)$ is isomorphic to R_K, where $K = Q(\sqrt{k})$. Let

$$\phi: G_2(k) \to R_K,\ K = Q(\sqrt{k}),$$

and

$$\phi\left(\begin{pmatrix} r & s \\ ks & r \end{pmatrix}\right) = r + s\sqrt{k}.$$

Then we prove that ϕ is an isomorphism. Indeed, we have, for $A, B \in G_2(k)$,

$$\phi(A \cdot B) = \phi(A) \cdot \phi(B) \quad \text{and} \quad \phi(A + B) = \phi(A) + \phi(B);$$

therefore, $G_2(k) \simeq R_K$, where $K = Q(\sqrt{k})$. The proof is complete.

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