A GENERALIZATION OF A RESULT OF SHANNON AND HORADAM

Dario Castellanos
Universidad de Carabobo, Valencia, Venezuela
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1. Introduction

In a recent note in this magazine [5] Professors A. G. Shannon and A. F. Horadam generalize a result proposed by Eisenstein [2] and solved by Lord [4] to the effect that

$$(1.1)\qquad L_n - \frac{(-1)^n}{L_n} - \frac{(-1)^n}{L_n} - \cdots = \alpha^n,$$

where L_n is the n^{th} Lucas number and α is the positive root of $x^2 - x - 1 = 0$.

They introduce the sequence $\{w_n\} \equiv \{w_n(a, b; p, q)\}$ defined by the initial conditions $w_0 = a$, $w_1 = b$, and the recurrence relation

$$(1.2)\qquad w_n = pw_{n-1} - qw_{n-2},\ n \geq 2,$$

where p and q are arbitrary integers.

They let $\alpha = (p + \sqrt{(p^2 - 4q)})/2$, $\beta = (p - \sqrt{(p^2 - 4q)})/2$, for $|\beta| < 1$, be the roots of

$$(1.3)\qquad x^2 - px + q = 0,$$

so that $\{w_n\}$ has the general term

$$(1.4)\qquad w_n = A\alpha^n + B\beta^n,$$

where

$$A = (b - a\beta)/d,\ B = (a\alpha - b)/d,\ AB = e/d^2;$$

$$e = pab - qa^2 - b^2,\ d = \alpha - \beta,\ p = \alpha + \beta,\ q = \alpha\beta.$$

They also let $Q_n = ABq^n$.

The Fibonacci sequence is

$$\{F_n\} \equiv \{w_n(0, 1; 1, -1)\},\ Q_n = (-1)^{n+1}/5;$$

the Lucas sequence is

$$\{L_n\} \equiv \{w_n(2, 1; 1, -1)\},\ Q_n = (-1)^n;$$

the Pell sequence is

$$\{P_n\} \equiv \{w_n(0, 1; 2, -1)\},\ Q_n = (-1)^n/8.$$

Shannon and Horadam's result is

$$(1.5)\qquad w_n - \frac{Q_n}{w_n} - \frac{Q_n}{w_n} - \cdots = A\alpha^n.$$

They establish this result by finding a general expression for the convergents of the continued fraction (1.5) and determining the limiting form with an appeal to some results of Khovanskii [3].

2. An Alternate Approach

Consider the identity

$$(2.1)\qquad \sqrt{s} - t = (s - t^2)/(2t + (\sqrt{s} - t)),$$

which gives at once the continued fraction (see [1])

$$\sqrt{s} = t + \frac{s - t^2}{2t\ +}\ \frac{s - t^2}{2t\ +}\ \frac{s - t^2}{2t\ +}\ \cdots . \tag{2.2}$$

In (2.2), replace s and t by $\frac{1}{4}t^2 - s$ and $\frac{1}{2}t$, respectively, to obtain

$$\sqrt{(\tfrac{1}{4}t^2 - s)} - \tfrac{1}{2}t = \frac{-s}{t\ +}\ \frac{-s}{t\ +}\ \frac{-s}{t\ +}\ \cdots,$$

or equivalently,

$$\sqrt{(\tfrac{1}{4}t^2 - s)} + \tfrac{1}{2}t = t - \frac{s}{t\ -}\ \frac{s}{t\ -}\ \frac{s}{t\ -}\ \cdots . \tag{2.3}$$

With the notation of Section 1, let $s = Q_n = AB(\alpha\beta)^n$, $t = w_n = A\alpha^n + B\beta^n$. Simple arithmetic shows that the left-hand side of (2.3) becomes $A\alpha^n$, and we find

$$A\alpha^n = w_n - \frac{Q_n}{w_n\ -}\ \frac{Q_n}{w_n\ -}\ \cdots, \tag{2.4}$$

which is the result of Shannon and Horadam.

Similarly, let $s = (-1)^{n+1}$, $t = 2F_n$, and recall that $F_n^2 + (-1)^n = F_{n-1}F_{n+1}$, and (2.3) gives

$$\sqrt{(F_{n-1}F_{n+1})} - F_n = \frac{(-1)^n}{2F_n\ +}\ \frac{(-1)^n}{2F_n\ +}\ \cdots . \tag{2.5}$$

As the reader no doubt knows, $\sqrt{(F_{n-1}F_{n+1})}$ is approximated by F_n, the approximation becoming better as n increases. The continued fraction in the right-hand side of (2.5) gives the error committed in the approximation.

Classes of expressions can be found by choosing suitable values of s and t. Especially interesting is the choice

$$t = a_1 w_{n_1}^{k_1} + a_2 w_{n_2}^{k_2} + \cdots + a_m w_{n_m}^{k_m},$$

where $k_1, k_2, \ldots, k_m, n_1, n_2, \ldots, n_m$ are arbitrary integers, $a_1, a_2, \ldots, a_m$ are arbitrary real numbers, and s is an arbitrary parameter.

Many other expressions can be found by giving appropriate values to s and t. It is left to the reader to discover them.

Acknowledgment

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References

1. D. Castellanos. "A Generalization of Binet's Formula and Some of Its Consequences." *Fibonacci Quarterly* 27.5 (1989):424-38. Equation (2.3) was discovered by the author. Joseph Ehrenfried Hofmann's *Geschichte der Mathematik* seems to indicate that a formula essentially equivalent to it was originally discovered by Michel Rolle in his *Mémoires de mathêmatiques et de physiques*, vol. 3 (Paris, 1692). C. D. Olds makes the claim, in his *Continued Fractions*, that the formula may have been known to Rafael Bombelli, a native of Bologna and a disciple of Girolamo Cardano, as far back as 1572.
2. M. Eisenstein. Problems B-530 and B-531. *Fibonacci Quarterly* 22 (1984):274.
3. A. N. Khovanskii. *The Application of Continued Fractions*. Tr. from Russian by Peter Wynn. Gronigen: Noordhoff, 1963.
4. G. Lord. Solutions to B-530 and B-531. *Fibonacci Quarterly* 23 (1985):280-81.
5. A. G. Shannon & A. F. Horadam. "Generalized Fibonacci Continued Fractions." *Fibonacci Quarterly* 26 (1988):219-23.

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