RECURRENT SEQUENCES INCLUDING N

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(Submitted December 1988)

Introduction

Suppose a (large) integer N is given and we wish to choose positive integers A, B such that

(a) the sequence $\{w_n\}$ defined by $w_1 = A$, $w_2 = B$, and $w_{n+2} = w_{n+1} + w_n$, $n \geq 1$, contains the integer N,
(b) $s = A + B$ is minimal.

What can be said about s in relation to N, and how are A and B to be found? We also consider some generalizations.

The case $N = 1{,}000{,}000$ was recently the subject of a problem in a popular computing magazine [1]. Obviously, for $N \geq 2$, $A = 1$, $B = N - 1$ is one pair satisfying (a) and so the problem does have a solution for each N. Also $s \geq 2$, and equality here holds whenever $N = F_k$, a Fibonacci number. Hence,

$$\liminf s = 2 \text{ as } N \to \infty.$$

In the opposite direction, we shall show that $s > \gamma\sqrt{N}$ for infinitely many N, but that for all sufficiently large N, $s < \gamma\sqrt{N} + 0(N^{-1/2})$, where $\gamma = 2/\sqrt{\alpha}$ and $\alpha = (1 + \sqrt{5})/2$. We shall also show how to select A and B for each N.

The Original Problem

Clearly, for a solution to the problem $A \geq B > 0$, for if $B > A$, then the pair $A_1 = B - A$, $B_1 = A$ would yield a smaller s. Starting from A, B, we then obtain, successively, A, B, $A + B$, ..., t, N and we now define, for each $t < N$, the sequence

$$t_0 = N,\ t_1 = t,\ t_{n+2} = t_n - t_{n+1},\ n \geq 0,$$

i.e., work backwards, so to speak, until we arrive at

$$t_k = A + B,\ t_{k+1} = B,\ t_{k+2} = A,\ t_{k+3} \leq 0.$$

Thus, the only choice at our disposal is t; k is then characterized by being the smallest integer for which $t_{k+3} \leq 0$, and our object is to choose t so as to minimize $s = t_k$.

Let α and β be the roots of $\theta^2 = \theta + 1$. Then $\alpha\beta = -1$, $\alpha + \beta = 1$, and

$$F_n = (\alpha^n - \beta^n)/(\alpha - \beta).$$

Then the roots of $\theta^2 = 1 - \theta$ are $-\alpha$ and $-\beta$, so that, for suitable constants c and d,

$$t_n = (-1)^n\{c\alpha^n + d\beta^n\}.$$

Using the initial conditions $t_0 = N$, $t_1 = t$, we then find that

$$(1) \qquad t_n = (-1)^n\{NF_{n-1} - tF_n\}.$$

Also, for $n > 0$,

$$(2) \qquad \alpha F_{n-1} - F_n = -\beta^{n-1} = (-1)^n\alpha^{-n+1},$$

and so

(3) $\quad (-1)^n\{\alpha F_{n-1} - F_n\} > 0.$

We now prove the following.

Theorem: Let

$$t_n = (-1)^n\{NF_{n-1} - tF_n\},$$

where $t_k = A + B$, $t_{k+1} = B$, $t_{k+2} = A$. Then $t = [n/\alpha]$ gives the smallest value for $t_k = A + B = s$ and

$$s < 2\sqrt{(N/\alpha)} \simeq 1.5723\sqrt{N}.$$

There are two cases. Suppose first that $N > \alpha t$. Then

$$t_n = (-1)^n t\{\alpha F_{n-1} - F_n\} + (-1)^n\{N - \alpha t\}F_{n-1} > (-1)^n\{N - \alpha t\}F_{n-1},$$

so t_n can be negative or zero only if n is odd. Thus, k must be even, and if $k = 2K$, then $t_{2K+1} > 0$, $t_{2K+3} \leq 0$. Thus, from (1)

$$\frac{F_{2K}}{F_{2K+1}} < \frac{t}{n} \leq \frac{F_{2K+2}}{F_{2K+3}}$$

and defining $\rho = N/\alpha - t > 0$, we have

$$\frac{F_{2K+3} - \alpha F_{2K+2}}{\alpha F_{2K+3}} \leq \frac{\rho}{N} < \frac{F_{2K+1} - \alpha F_{2K}}{\alpha F_{2K+1}}$$

i.e., in view of (2),

(4) $\quad \alpha^{2K+1}F_{2K+1} < N/\rho \leq \alpha^{2K+3}F_{2K+3},$

whence,

$$\alpha^{4K+2} + 1 = \alpha^{2K+1}(\alpha^{2K+1} - \beta^{2K+1}) < N\sqrt{5}/\rho$$
$$\leq \alpha^{2K+3}(\alpha^{2K+3} - \beta^{2K+3}) = \alpha^{4K+6} + 1;$$

so

(5) $\quad \alpha^{4K+2} < N\sqrt{5}/\rho - 1 \leq \alpha^{4K+6}.$

Also, in this case,

(6) $$s = t_{2K} = NF_{2K-1} - tF_{2K}$$
$$= N(F_{2K-1} - F_{2K}/\alpha) + \rho F_{2K}$$
$$= N/\alpha^{2K} + \rho F_{2K} = \xi + \eta, \text{ say.}$$

Of these two terms, ξ is always the larger; in fact, from (4), we have

(7) $$\frac{\alpha F_{2K+1}}{F_{2K}} < \frac{\xi}{\eta} = \frac{N}{\rho\alpha^{2K}F_{2K}} \leq \frac{\alpha^3 F_{2K+3}}{F_{2K}},$$

whence

(8) $\quad \alpha^2 < \xi/\eta \leq \alpha^6 + 2|\beta|^{2K-3}/F_{2K}.$

We now show that, for all $t < N/\alpha$, $t = [N/\alpha]$ gives the smallest value for s. For, let $t = [N/\alpha]$ and $t' < t$ be any other integer, yielding, respectively, ρ, K, ξ, η, s and ρ', K', ξ', η', and s'. Then $t' \leq t - 1$, whence $\rho' \geq \rho + 1$ and, in view of (5), $K' \leq K$. If $K' = K$, then $\xi' = \xi$ and $\eta' > \eta$, which gives $s' > s$, whereas, if $K' < K$, then

$$s' = \xi' + \eta' > \xi' \geq \alpha^2\xi = (\alpha + 1)\xi > \xi + \eta = s,$$

in view of (8). Moreover, using (7), we see that

$$\begin{aligned}\frac{s^2}{N} &= \frac{(\xi+\eta)^2}{N} = \frac{\xi\eta}{N}\left\{\frac{\xi}{\eta} + 2 + \frac{\eta}{\xi}\right\} \le \frac{\rho F_{2K}}{\alpha^2}\left\{\frac{\alpha^3 F_{2K+3}}{F_2} + 2 + \frac{F_{2K}}{\alpha^3 F_{2K+3}}\right\}\\ &= \frac{\rho}{\alpha^{2K+3}F_{2K+3}}(\alpha^3 F_{2K+3} + F_{2K})^2\\ &= \frac{\rho(\alpha-\beta)}{\alpha^{4K+6}+1}\left\{\frac{\alpha^3(\alpha^{2K+3}-\beta^{2K+3}) + (\alpha^{2K}-\beta^{2K})}{(\alpha-\beta)}\right\}^2\\ &= \frac{\rho\alpha^{4K+6}}{\alpha^{4K+6}+1}\cdot\frac{(\alpha^3-\beta^3)^2}{(\alpha-\beta)} < 4\rho\sqrt{5}.\end{aligned}$$

Thus,

(9) $\quad s < 2N^{1/2}\rho^{1/2}5^{1/4}$.

The case in which $N < \alpha t$ is entirely similar. Suppressing the details we find that k must be odd, and if $k = 2M - 1$, then with $\sigma = t - N/\alpha$, we obtain

(4′) $\quad \alpha^{2M}F_{2M} < N/\sigma \le \alpha^{2M+2}F_{2M+2}$,

(5′) $\quad \alpha^{4M} < N\sqrt{5}/\sigma + 1 \le \alpha^{4M+4}$,

(6′) $\quad s = N/\alpha^{2M-1} + \sigma F_{2M-1} = \xi + \eta$, say.

(7′) $\quad \dfrac{F_{2M}}{F_{2M-1}} < \dfrac{\xi}{\eta} = \dfrac{N}{\sigma\alpha^{2M-1}F_{2M-1}} \le \dfrac{\alpha^3 F_{2M+2}}{F_{2M-1}}$,

(8′) $\quad \alpha^2 - \beta^{4M-3}\sqrt{5} < \xi/\eta < \alpha^6$.

For all sufficiently large N,

(9′) $\quad s < 2N^{1/2}\sigma^{1/2}5^{1/4} + O(N^{-1/2})$.

At this stage we may immediately make the observation that, for any N, one of σ and ρ lies below 1/2, and so (9) and (9′) immediately give an upper bound of $(2N\sqrt{5})^{1/2} + O(N^{-1/2})$ or approximately $2\cdot 115N^{1/2}$. It is, however, possible to improve this.

Let us suppose that $\rho/\sigma = \alpha^{-2\theta}$, so that

(10) $\quad \rho = 1/(1+\alpha^{2\theta})$ and $\sigma = \alpha^{2\theta}/(1+\alpha^{2\theta})$,

since $\sigma + \rho = 1$. Then, if $\theta \ge 1 - 1/N$, i.e., ρ is small, we use the inequality (9), and if $\theta \le -1 + 1/N$, i.e., σ is small, we use the inequality (9′) and, in either case, obtain

(11) $\quad s < 2N^{1/2}5^{1/4}/(1+\alpha^2)^{1/2} + O(N^{-1/2}) = \gamma N^{1/2} + O(N^{-1/2})$,

as required. The remaining case is $|\theta| < 1 - 1/N$. Let $N\sqrt{5}/\rho - 1 = \alpha^\lambda$, and let $N\sqrt{5}/\sigma + 1 = \alpha^\mu$. Then a little manipulation yields

$$2\theta > \lambda - \mu > 2\theta - 1/N,$$

and so, certainly, $|\lambda - \mu| < 2$. Then we have, from (5), that $\alpha^\lambda > \alpha^{4K+2}$, i.e., $\lambda > 4K + 2$ and, from (5′), that $4M + 4 > \mu$. Since $\mu + 2 > \lambda$, $4M + 6 > 4K + 2$ and so $M \ge K$. Similarly, we find that $M \le K - 1$, and so all in all $M = K$ or $K - 1$; in other words, the values of k obtained from ρ or σ differ by exactly one. It is easy to see that whichever is the larger value would give the sharper bound for s, but there is no a priori way to determine which does indeed give the larger k. If it is $2K$, then we can improve the bound given by (9), by observing that

$$\lambda < \mu + 2\theta < 4M + 4 + 2\theta,$$

and so the upper bound for ξ/η given by (8) can be improved to $\alpha^{4+2\theta} + O(1/N)$ and then the same argument which led to (9) now leads to

$$\frac{s^2}{N} < \frac{\rho(\alpha^{2+\theta} + \alpha^{-2-\theta})^2}{(\alpha - \beta)} + O(1/N) = \frac{(\alpha^{2\theta} + \alpha^{-2-\theta})^{2\theta}}{(\alpha^2 + 1)\sqrt{5}} + O(1/N)$$
$$= f(\theta) + O(1/N), \text{ say.}$$

In the same way, it is possible to improve the bound if the larger value is given by $2M - 1$, and the corresponding bound for s^2/N is just $f(-\theta) + O(1/N)$. Since we do not know which of these will apply, we must take the larger one, i.e., $g(\theta) = \max\{f(\theta), f(-\theta)\}$. It is quite simple to see that $f(\theta)$ is an increasing function of θ and so the worst case arises from $(1 - 1/N)$, the upper bound for $|\theta|$, giving

$$s^2/N < 4/\alpha + O(1/N),$$

yielding (11) again. This concludes the proof of the theorem.

Now, we show that this bound cannot be reduced. Choosing $N = F_{2n+1}F_{2n+2}$, we find that

$$[N/\alpha] = (\alpha^{4n+2} + \beta^{4n+2} - 3)/5, \quad \rho = (\alpha + \beta^{4n+3})/\sqrt{5},$$
$$\sigma = -(\beta + \beta^{4n+3})/\sqrt{5}, \quad \lambda = 4n + 2, \quad \mu = 4n + 4,$$

and so $K = n - 1$ and $M = n$. Therefore, it follows that the latter gives the larger value for k, and that, in view of (9′),

$$\begin{aligned} s &= N\alpha^{1-2n} + \rho F_{2n-1} \\ &= \frac{(\alpha^{2n+1} - \beta^{2n+1})(\alpha^{2n+2} - \beta^{2n+2})}{5\alpha^{2n-1}} - \frac{(\beta + \beta^{4n+3})(\alpha^{2n-1} - \beta^{2n-1})}{5} \\ &= \frac{1}{5}\{\alpha^{2n+4} + \beta^{2n-2} - \beta^{2n} - \beta^{6n+4} + \alpha^{2n-2} + \beta^{2n} + \beta^{2n+4} + \beta^{6n+4}\} \\ &= \frac{1}{5}(\alpha^{2n+1} - \beta^{2n+1})(\alpha^3 - \beta^3) = F_3F_{2n+1} = 2F_{2n+1}, \end{aligned}$$

and now

$$\frac{s^2}{N} = \frac{4F_{2n+1}}{F_{2n+2}} = \frac{4(\alpha^{2n+1} - \beta^{2n+1})}{(\alpha^{2n+2} - \beta^{2n+2})} > \frac{4}{\alpha}.$$

This concludes the discussion of the original problem.

Generalizations

Several generalizations are now possible. the simplest of these consists of choosing a given integer $a \geq 1$ and replacing the original relations by

(a1) the sequence $\{w_n\}$ defined by $w_1 = A$, $w_2 = B$, and $w_{n+2} = aw_{n+1} + w_n$, $n \geq 1$, contains the integer N,
(b1) $s = aB + A$ is minimal.

This creates but minor changes in the working above. We now let $\alpha > 0$ and $\beta < 0$ be the roots of $\theta^2 = a\theta + 1$ and then $\alpha\beta = -1$, $\alpha + \beta = a$, $\alpha - \beta = (a^2 + 4)^{1/2}$. We define F_n as before *in terms of* α *and* β, although, of course, F_n will no longer be the Fibonacci number. The effects of this are to replace $\sqrt{5}$ wherever it occurs by the new value of $\alpha - \beta$, and to replace the number $2 = F_3$ in formulas (9), (9′), and (11) and in the value of γ, by $a^2 + 1$. The form of the result remains identical, with

$$\gamma = (a^2 + 1)/\sqrt{\alpha} \quad \text{and} \quad \alpha = (a + (a^2 + 4)^{1/2})/2.$$

The details are omitted.

The next generalization we consider consists of replacing the original relations by (a2) the sequence $\{w_n\}$ defined by $w_1 = A$, $w_2 = B$, and $w_{n+2} = aw_{n+1} - w_n$, $n \geq 1$, contains the integer N,

(b2) $s = aB - A > 0$ is minimal.

Here the integer a cannot be 1, otherwise any such sequence would contain only six distinct numbers, or 2, otherwise the problem becomes trivial since we could always take $w_1 = 1$, $w_2 = 2$, and then $w_N = N$ with $s = 3$. So we assume that $a \geq 3$. We now let the roots of $\theta^2 = a\theta - 1$ be

$$\alpha = (a + (a^2 - 4)^{1/2})/2 \quad \text{and} \quad \beta = (a - (a^2 - 4)^{1/2})/2,$$

and then $\alpha\beta = 1$, $\alpha + \beta = a$, with

$$0 < \beta < 1 < \alpha \quad \text{and} \quad \alpha - \beta = \alpha = (a^2 - 4)^{1/2}.$$

Again we let $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, and proceeding as before we let the integer in the sequence before N be t, and obtain A, B, $aB - A$, ..., t, N, and so, if $t_0 = N$, $t_1 = t$, $t_{n+2} = at_{n+1} - t_n$, we get a reverse sequence where

(12) $\quad t_n = tF_n - NF_{n-1},$

(13) $\quad F_n - \alpha F_{n-1} = \beta^{n-1} > 0,$

(14) $\quad t_n = -(N - t\alpha)F_{n-1} + t\beta^{n-1}.$

What happens now depends on the sign of $(N - t\alpha)$.

Case I. $N > t\alpha$. Then, eventually, t_n becomes negative, and we find that

$$s = F_k,\ F_{k+1} = B,\ F_{k+2} = A, \text{ and } F_{k+3} \leq 0.$$

All this parallels the previous work with only minor differences, and if $\rho = N/\alpha - t$, then we find that

(15) $\quad \alpha^{2k+6} \geq 1 + N(\alpha - \beta)/\rho > \alpha^{2k+4},$

(16)
$$\begin{aligned} s = t_k &= tF_k - NF_{k-1} \\ &= N(F_k/\alpha - F_{k-1}) - \rho F_k \\ &= N/\alpha^k - \rho F_k = \xi - \eta, \text{ say.} \end{aligned}$$

(17) $\quad \alpha^4 < \xi/\eta < \alpha^6 + O(1/N).$

Unfortunately, it is no longer necessarily the case that $s' > s$ whenever $t' < t = [N/\alpha]$. For we have $t' \leq t - 1$, whence $\rho' \geq \rho + 1$, and so, in view of (15), $k' \leq k$. Now, if indeed $k' < k$, then $s' > s$, for

$$s' = \xi' - \eta' > \xi'(1 - 1/\alpha^4) \geq \alpha\xi(1 - 1/\alpha^4) > \xi > \xi - \eta = s.$$

However, if $k' = k$, then $s' < s$, since now $\rho' > \rho$. Although this is true, we shall see presently that it causes no problems, for then $\rho' > 1$, and in such a case a choice with $t > N/\alpha$ would always yield a smaller s. In any event, we obtain a result analogous to (9),

(18) $\quad s < (a^2 - 1)N^{1/2}\rho^{1/2}(a^2 - 4)^{1/4} + O(N^{-1/2}).$

Case II. $N < t\alpha$, is entirely different. Let $t = N/\alpha + \sigma$. Then

$$t_n = t\beta^{n-1} + \sigma\alpha F_{n-1}$$

is positive for all $n > 0$, and we now need to choose $k = K$ to minimize $s = t_k$. Then $t_K \leq t_{K+1}$ gives, in view of (12),

$$N(F_K - F_{K-1}) \leq (F_{K+1} - F_K)t = (F_{K+1} - F_K)(N/\alpha + \sigma)$$

and so, using (13),

$$(F_{K+1} - F_K)\sigma \geq N(\beta^K - \beta^{K+1})$$

and so

$$(1 - \beta)(\alpha^{K+1} + \beta^K)\sigma \geq N(\alpha - \beta)(1 - \beta)\beta^K$$

which, together with a similar inequality obtained from $t_K \leq t_{K-1}$ yields

(19) $$\alpha^{2K-1} \leq N(\alpha - \beta)/\sigma - 1 \leq \alpha^{2K+1},$$

and then

$$\begin{aligned} s = t_K &= tF_K - NF_{K-1} \\ &= (N/\alpha + \sigma)F_K - NF_{K-1} \\ &= N/\alpha^K + \sigma F_K = \xi + \eta, \text{ say.} \end{aligned}$$

In this case, it is clear that the smallest s is provided by taking σ as small as possible, and we find, using (19), that the ratio η/ξ lies between α and $(\alpha^{2K} - 1)/(\alpha^{2K+1} + 1) < 1/\alpha$, and so we obtain, as before,

$$\begin{aligned} \frac{s^2}{N} &= \frac{(\xi + \eta)^2}{N} = \frac{\xi\eta}{N}\left\{\frac{\xi}{\eta} + 2 + \frac{\eta}{\xi}\right\} \\ &\leq \frac{\sigma F_K}{\alpha^K}\left\{\frac{\alpha^{2K} - 1}{\alpha^{2K+1} + 1} + 2 + \frac{\alpha^{2K+1} + 1}{\alpha^{2K} + 1}\right\} \\ &= \frac{\sigma\alpha^{2K+1}(\ \ + 1)}{(\alpha^{2K+1} + 1)(\alpha - 1)} < \frac{\sigma(\alpha + 1)}{(\alpha - 1)}. \end{aligned}$$

Thus,

(20) $$s < N^{1/2}\sigma^{1/2}\left\{\frac{1 + \beta}{1 - \beta}\right\}^{1/2}$$

and this bound is much better than that provided by (18) unless ρ is extremely small, certainly less than 1. This justifies our earlier remark that we need only consider the smallest value of ρ. Since, at any rate, we can always take $\sigma < 1$ in (20), we obtain immediately

$$s < N^{1/2}\left\{\frac{1 + \beta}{1 + \beta}\right\}^{1/2}.$$

This can be improved slightly, and we prove that $s < N^{1/2}\,\delta$, where

$$\delta^2 = \frac{1 + \beta}{1 - \beta}\,\frac{1}{1 + \beta^3}.$$

As before, we define θ by $\rho/\sigma = \alpha^{-2\theta}$ obtaining (10), and define λ and μ by

$$N(\alpha - \beta)/\rho - 1 = \alpha^\lambda \quad\text{and}\quad N(\alpha - \beta)/\sigma + 1 = \alpha^\mu,$$

whence

$$2\theta < \lambda - \mu < 2\theta + 1/N.$$

If now $\theta \leq 3/2$, then $\sigma \leq (1 + \beta^3)^{-1/2}$ and then (20) gives the required result, whereas if $\theta > 5/2 - 1/N$, then we find that

$$\rho^2 < \beta^5/(1 + \beta^5) + O(1/N)$$

and then, using (18), we find that

$$\frac{s^2}{N} < \frac{(\alpha^3 - \beta^3)^2}{\alpha - \beta}\,\frac{\beta^5}{1 + \beta^5} + O(1/N),$$

and since $\beta < 1$, the result easily follows. The remaining case is where

$$3 < \lambda - \mu < 5$$

and then, in view of (15) and (19), we find that $2k < \lambda - 4$ and $2K \geq \mu - 1$, whence

$$2(K - k) > \mu - \lambda + 3 > -2,$$

and so, since both k and K are integers, $K \geq k$. Thus, from (16) and (19), we find that

$$s < N/\alpha^k \leq N/\alpha^K < N^{1/2}(1 - \beta^2)^{-1/2} + O(N^{-1/2})$$

and again the result follows.

The following example shows that the result is best possible. Let $N = (F_{n+1} - F_n)L$, where the integer L is to be chosen later. Then

$$N/\alpha = \frac{L}{\alpha - \beta}\{\alpha^n - \alpha^{n-1} - \beta^{n+2} + \beta^{n+1}\}$$
$$= (F_n - F_{n-1})L - L\beta^n(1 - \beta),$$

and so

$$[N/\alpha] = (F_n - F_{n-1})L - 1,$$

provided that $L\beta^n(1 - \beta) < 1$. It is easily seen that this latter condition is equivalent to $L \leq F_{n+1} + F_n$, so we let $L = F_{n+1} + F_n - x$, where $x \geq 0$ is to be chosen later. If we now take $t = (F_n - F_{n-1})L = [N/\alpha]$, then

$$t_r = (F_{n-r+1} - F_{n-r})L,$$

so the least $t_r = t_n = t_{n+1} = L$. On the other hand, if $t = [N/\alpha] - 1$, then

$$t_r = (F_{n-r+1} - F_{n-r})L - F_r,$$

so

$$t_n = L - F_n = F_{n+1} - x,$$
$$t_{n+1} = L - F_{n+1} = F_n - x,$$

and $$t_{n+2} = F_{n-1} - x(a - 1).$$

Now, if we choose x to be the least integer $\geq F_{n-1}/(a - 1)$, then we find that $k = n - 1$, and the value of t_k exceeds L, the value given for s by the other choice. Hence, for such an N, we obtain

$$\frac{s^2}{N} = \frac{F_{n+1} + F_n - x}{F_{n+1} - F_n} = \frac{(a - 1)(F_{n+1} + F_n) - F_{n-1}}{(a - 1)(F_{n+1} - F_n)} + O(1)$$
$$= \frac{aF_{n+1} - F_n}{(a - 1)(F_{n+1} - F_n)} + O(1)$$
$$= \frac{(1 + \beta^2) - \beta^2}{(1 - \beta + \beta^2)(1 - \beta)} + O(1)$$
$$= \frac{1 + \beta}{1 - \beta}\,\frac{1}{1 + \beta^3} + O(1) = \delta^2 + O(1), \text{ say.}$$

Thus, letting $n \to \infty$, we find that $s < N^{1/2}\delta + O(N^{-1/2})$.

Reference

1. "Leisure Lines." *Personal Computer World* *11.3* (1988):211.

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