SOLUTIONS OF FERMAT'S LAST EQUATION IN TERMS OF WRIGHT'S HYPERGEOMETRIC FUNCTION

Allen R. Miller
Naval Research Laboratory, Washington, D.C. 20375-5000
(Submitted January 1989)

Introduction

In this paper we study a problem related to Fermat's last theorem. Suppose that X, Y, and Z are positive numbers where

(1) $$X^a + Y^a = Z^a.$$

We show that we can solve this equation for a; that is, we find a unique

$$a = a(X, Y, Z)$$

in closed form. The method of solution is rather elementary, and we employ Wright's generalized hypergeometric function in one variable [1], as defined below:

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\ (\beta_1, B_1), \ldots, (\beta_q, B_q);\end{matrix}\; z\right] \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i + A_i n)}{\prod_{i=1}^{q}\Gamma(\beta_i + B_i n)} \frac{z^n}{n!}.$$

When $p = q = 1$, we see that

(2) $${}_1\Psi_1\left[\begin{matrix}(\alpha, A); \\ (\beta, B);\end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + An)}{\Gamma(\beta + Bn)} \frac{z^n}{n!},$$

which is a generalization of the confluent hypergeometric function ${}_1F_1[\alpha; \beta; z]$.

An Equivalent Form of Equation (1)

In Equation (1), the case $X = Y$ is not interesting since, clearly,

$$a = \frac{\ln(1/2)}{\ln(X/Z)}.$$

Therefore, we shall assume, without loss of generality, that

$$Z > Y > X > 0,$$

and write Equation (1) as

$$e^{a\ln(X/Z)} + e^{a\ln(Y/Z)} - 1 = 0.$$

Now, making the transformation

(3) $$e^{a\ln(Y/Z)} \equiv y,$$

we obtain

$$y^{\frac{\ln(X/Z)}{\ln(Y/Z)}} + y - 1 = 0,$$

and since

$$\frac{\ln(X/Z)}{\ln(Y/Z)} = \frac{\ln(Z/X)}{\ln(Z/Y)} > 1,$$

we arrive at

(4) $$y^{\frac{\ln(Z/X)}{\ln(Z/Y)}} + y - 1 = 0.$$

Equation (4) is then equivalent to Equation (1), and our aim is to solve this equation for y, thereby obtaining a. We note that it is not difficult to verify that Equation (4) has a unique positive root y in the interval (1/2, 1).

Solution of Equation (4)

In 1915, Mellin [2, 3] investigated certain transform integrals named after him in connection with his study of the trinomial equation

$$y^N + xy^P - 1 = 0, \quad N > P, \tag{5}$$

where x is a real number and N, P are positive integers. Mellin showed that, for appropriately bounded x, a positive root of Equation (5) is given by

$$y = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)x^{-z}\,dz, \quad 0 < c < 1/P, \tag{6}$$

where

$$F(z) = \frac{\Gamma(z)\Gamma\left(\frac{1}{N} - \frac{P}{N}z\right)}{N\Gamma\left[1 + \frac{1}{N} + \left(1 - \frac{P}{N}\right)z\right]}$$

and

$$|x| < (P/N)^{-P/N}(1 - P/N)^{P/N-1} \le 2. \tag{7}$$

The inverse Mellin transform, Equation (6), is evaluated by choosing an appropriate closed contour and using residue integration to find that

$$y = \frac{1}{N}\sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{N} + \frac{P}{N}n\right)}{\Gamma\left[1 + \frac{1}{N} + \left(\frac{P}{N} - 1\right)n\right]} \frac{(-x)^n}{n!}. \tag{8}$$

Under the condition shown in Equation (7), Mellin, in fact, found all of the roots of Equation (5). However, suppose we relax the restriction that N and P are positive integers. Instead, let N and P be positive numbers. We then observe that Equation (8) gives *a fortiori* a positive root of Equation (5) for positive numbers N and P. Further, without loss of generality, we set $P = 1$, $N = w$. Then, using the Wright function defined by Equation (2), we arrive at the following. The unique positive root of the transcendental equation

$$y^{\omega} + xy - 1 = 0, \quad \omega > 1, \tag{9}$$

where

$$|x| < \omega/(\omega - 1)^{1-1/\omega}$$

is given by

$$y = \frac{1}{\omega}\, {}_1\Psi_1\left[\begin{matrix}\left(\frac{1}{\omega}, \frac{1}{\omega}\right) & ; \\ \left(\frac{1}{\omega} + 1, \frac{1}{\omega} - 1\right); & -x\end{matrix}\right]. \tag{10}$$

We observe that for any $|x| < \infty$, Equation (9) has a unique positive root y. Equations (9) and (10) may also be obtained from Equation (30) on page 713 of [4].

Let us now apply the latter result to Equation (4). On setting

$$x = 1, \quad \omega^{-1} = \frac{\ln(Z/Y)}{\ln(Z/X)} \equiv \lambda,$$

and noting that $1 < \omega/(\omega - 1)^{1-1/\omega}$, we find

(11) $$y = \lambda\, {}_1\Psi_1\left[\begin{matrix}(\lambda,\ \lambda) & ; \\ & & -1 \\ (\lambda + 1,\ \lambda - 1); & \end{matrix}\right],\ 0 < \lambda < 1.$$

Solution of Equation (1)

We now solve Equation (1) for a. From the transformation Equation (3), we see that

(12) $$a \ln(Y/Z) = \ln y.$$

Then, using Equation (11), we arrive at the following. If $Z > Y > X > 0$ are such that

$$X^a + Y^a = Z^a,$$

then

(13) $$a = \frac{\ln\left\{\lambda\, {}_1\Psi_1\left[\begin{matrix}(\lambda,\ \lambda) & ; \\ & & -1 \\ (\lambda + 1,\ \lambda - 1); & \end{matrix}\right]\right\}}{\ln(Y/Z)},$$

where

(14) $$\lambda \equiv \frac{\ln(Z/Y)}{\ln(Z/X)},\ 0 < \lambda < 1.$$

We now prove the following. Consider for $X < Y$, $M \geq 1$, the diophantine equation

$$X^M + Y^M = Z^M.$$

Then the positive integers X, Y, and Z must satisfy

(15) $$X^\lambda Y^{-1} Z^{1-\lambda} = 1,$$

where λ is an irrational number such that $0 < \lambda < 1$.

From Equation (12) we have

(16) $$(Y/Z)^M = y,$$

so that y is a rational number in the interval $1/2 < y < 1$ as we noted earlier. If λ is rational, there exist relatively prime integers s and t such that

$$\lambda = \omega^{-1} = s/t.$$

Hence, y is the unique positive root of

$$y^{t/s} + y - 1 = 0.$$

Now, since $\lambda < 1$, then $s < t$, and we obtain the polynomial equation of degree t with integer coefficients:

$$y^t + (-1)^s y^s + \cdots + 1 = 0.$$

The only positive rational root that this equation may have is $y = 1$ (see [5], p. 67). But $y < 1$, so the assumption that λ is rational leads to a contradiction. We have then that λ is irrational, and Equation (15) follows from Equation (14). This proves our result. W. P. Wardlaw has given another proof that λ is irrational in [6].

The Wright function ${}_1\Psi_1$ appearing in Equation (13) depends only on the parameter λ. Thus, for brevity, we define

$$\Psi(\lambda) \equiv {}_1\Psi_1\left[\begin{matrix}(\lambda,\ \lambda) & ; \\ & & -1 \\ (\lambda + 1,\ \lambda - 1); & \end{matrix}\right],\ 0 < \lambda < 1.$$

From our previous result, we see that, if Fermat's theorem* is false, then there exist positive integers $X < Y < Z$ such that λ is irrational.

Therefore, Fermat's theorem is false if and only if there exist positive integers $Y < Z$, $M > 2$, and an irrational number λ $(0 < \lambda < 1)$ such that

$$(Y/Z)^M = \lambda\Psi(\lambda).$$

Thus, Fermat's conjecture may be posed as a problem involving the special function $\lambda\Psi(\lambda)$. We remark that recently, Fermat's conjecture has been given in combinatorial form [7].

Some Elementary Properties of $\lambda\Psi(\lambda)$

Although the series representation for $\lambda\Psi(\lambda)$, which follows below in Equation (17), does not converge for $\lambda = 0, 1$, it is natural to define

$$\lambda\Psi(\lambda)\Big|_{\lambda=1} = 1/2, \quad \lambda\Psi(\lambda)\Big|_{\lambda=0} = 1.$$

Using this definition, we give a brief table of values for $\lambda\Psi(\lambda)$, which is correct to five significant figures:

λ	$\lambda\Psi(\lambda)$	λ	$\lambda\Psi(\lambda)$
0.0	1.00000	0.6	0.58768
0.1	0.83508	0.7	0.56152
0.2	0.75488	0.8	0.53860
0.3	0.69814	0.9	0.51825
0.4	0.65404	1.0	0.50000
0.5	0.61803		

Observe that we may write the inverse relation

$$\lambda = \ln \lambda\Psi(\lambda)/\ln[1 - \lambda\Psi(\lambda)].$$

Note also that when $\lambda = 1/2$, $\omega = 2$ and Equation (9) becomes $y^2 + y - 1 = 0$, whose positive root is $(-1 + \sqrt{5})/2$.

The following series representations for $\lambda\Psi(\lambda)$, $0 < \lambda < 1$ may easily be derived from the first one below:

$$(17) \qquad \lambda\, {}_1\Psi_1\begin{bmatrix}(\lambda,\ \lambda) & ; \\ & -1 \\ (\lambda + 1,\ \lambda - 1); & \end{bmatrix} = \lambda \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(\lambda + \lambda n)}{\Gamma(\lambda + 1 + (\lambda - 1)n)}$$

$$(18) \qquad = \frac{\lambda}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(1 - \lambda)n - 1} \sin[\pi(1 - \lambda)n] B(\lambda n,\ n - \lambda n)$$

$$(19) \qquad = 1 - \lambda \sum_{n=0}^{\infty} (-1)^n {}_2F_1[-n,\ (1 - \lambda)(n + 2);\ 2;\ 1]$$

$$(20) \qquad = 1 + \lambda \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \binom{\lambda(1 + n) - 1}{n - 1}.$$

Equation (18) follows from Equation (17) by using

$$\Gamma(z)\Gamma(-z) = -\pi/z \sin \pi z;$$

$B(x, y)$ is the beta function. Equation (19) follows from Equation (17) by using Gauss's theorem for ${}_2F_1[a, b; c; 1]$. Equation (20) follows from Equation (17) by using

$$\binom{\alpha}{m} = \Gamma(1 + \alpha)/m!\Gamma(1 + \alpha - m).$$

---

*Fermat's theorem states that there are no integers $x, y, z > 0$, $n > 2$ such that $x^n + y^n = z^n$.

Equation (20), for $1/\lambda$ an integer greater than one, is due to Lagrange ([2], p. 56).

Conclusion

The equation $X^a + Y^a = Z^a$ has been solved for a as a function of X, Y, and Z in terms of a Wright function ${}_1\Psi_1$ with negative unit argument. An equivalent form of Fermat's last theorem has been given using this function. Further, some elementary properties of ${}_1\Psi_1$ have been stated.

References

1. H. M. Srivastava & H. L. Manocha. *A Treatise on Generating Functions.* Halsted Press, 1984.
2. M. G. Belardinelli. "Résolution analytique des equations algebraiques generales." *Mémorial des Sciences Mathematiques*, Fascicule 145. Gauthiers-Villars, 1960.
3. H. Hochstadt. *The Functions of Mathematical Physics.* Wiley, 1971.
4. A. P. Prudnikov, Yu. A. Brychkov, & O. I. Marichev. *Integrals and Series*, Vol. 1. Gordon and Breach, 1986.
5. S. Borofsky. *Elementary Theory of Equations.* Macmillan, 1950.
6. W. P. Wardlaw, personal communication.
7. W. V. Quine. "Fermat's Last Theorem in Combinatorial Form." *Amer. Math. Monthly* 95 (1988):636.

*****