FIBONACCI NUMBERS ARE NOT CONTEXT-FREE

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(Submitted February 1989)

The Fibonacci numbers, given by the recurrence relation

$$F(n + 2) = F(n + 1) + F(n),\ F(1) = 1,\ F(2) = 1,$$

are considered to be written in base b, so "trailing zeros" correspond exactly to "factors of b." From [4], Theorem 5, page 527, it follows that, for any prime p, there exists n s.t. $F(k \times n) \equiv 0 \pmod{p^i}$ for positive i and k. The existence of j s.t. $F(j) \equiv 0 \pmod{b^i}$, for arbitrary positive b, follows by applying the above to the prime factoring of b and choosing j to be the least common multiple of the n. Thus, in any base, there exist Fibonacci numbers with arbitrarily many trailing zeros.

In the proof of this same theorem [4], it is established for any prime p that, if $F(n)$ is the first term $\equiv 0 \pmod{p^e}$ but $\not\equiv 0 \pmod{p^{e+1}}$, then $F(p \times n)$ is the first term $\equiv 0 \pmod{p^{e+1}}$, also $F(p \times n) \not\equiv 0 \pmod{p^{e+2}}$.

This establishes, for each prime base p, a lower bound on n which increases exponentially with the number of trailing zeros in $F(n)$ base p. This bound generalizes to composite bases because when $F(n)$ has e trailing zeros in base b it must also have e trailing zeros in all bases p, where p is a prime factor of b. Specifically, there is some constant k such that, for all sufficiently large n,

$$TZ(F(n)) < k \times \log(n),$$

where $TZ(x)$ is the number of trailing zeros in x.

Since the Fibonacci sequence is asymptotically exponential, there is some constant c s.t. $n < c \times |F(n)|$, where $|F(n)|$ denotes the *length* of $F(n)$ as a string, i.e., the number of digits in $F(n)$ in base b. Combining these, and adjusting k to also account for c, gives

$$(1) \qquad TZ(F(n)) < k \times \log(|F(n)|).$$

These facts can be used to show that the Fibonacci numbers do not form a *context-free* set. A set of strings is context-free iff it is the set generated by some context-free grammar or, equivalently, a set of strings is context-free iff it is the set recognized by some pushdown automaton. Ogden's Lemma, stated below, gives a property true of all context-free sets, and is used in Lemma 1 to show a set of strings closely related to the Fibonacci numbers to be not context-free.

Ogden's Lemma [2]: Let Q be a context-free set. Then there is a constant j such that, if α is any string in Q and we mark any j or more positions of α "distinguished," then we can write $\alpha = uvwxy$, such that:

1) v and x together have at least one distinguished position,
2) vwx has at most n distinguished positions, and
3) for all $i \geq 0$, uv^iwx^iy is in Q.

Lemma 1: Let Q be the set of strings such that the members of Q are the Fibonacci numbers written in base b with a new symbol "#" inserted immediately following the last nonzero digit. The set Q is not context-free.

Proof: The proof is by contradiction. Assume that Q is context-free.

Let j be the number of "distinguished" positions required for Ogden's Lemma (see [2] for a description of Ogden's Lemma). Since we know there are

Fibonacci numbers with arbitrarily many trailing zeros, let α be a member of Q corresponding to a Fibonacci number with at least j trailing zeros. The trailing zeros, which follow the "#," are used as the distinguished positions for purposes of Ogden's Lemma.

Applying Ogden's Lemma, α may be partitioned as follows:

$$\alpha = uvwxy,$$

where x contains at least one of the trailing zeros. Further, for all $i \geq 0$, $\beta_i = uv^iwx^iy$ must also be in the set Q, and thus correspond to some Fibonacci number satisfying (1).

If x contained the "#," then clearly β_2 would contain two "#" symbols and, thus, could not be a member of Q. Therefore, x contains only "0"s, so β_i has at least $j + i - 1$ trailing zeros.

Since v and x together can be no longer than α, then β_i can be no more than i times as long as α: So $|\beta_i| \leq i \times |\alpha|$. Applying (1) to these bounds gives:

$$j + i - 1 < k \times \log(i \times |\alpha|).$$

Choosing $i = 2k^2|\alpha| + 1$ produces a contradiction. □

Theorem: For all integers $b \geq 2$, the set of Fibonacci numbers in base b, considered as strings over the alphabet 0, 1, ..., $b - 1$, is not context-free.

Proof: Assume M is a pushdown automaton (PDA) recognizing the set of Fibonacci numbers. We modify the finite control to give another PDA M', recognizing the set Q, thus contradicting Lemma 4. An informal description of M' follows.

M' contains a copy of the machine M, plus additional logic in the finite control to filter the input and pass it to this internal copy of M. M' accepts only when this internal M accepts the string passed to it.

behaves as follows:

- M' rejects if the input does not contain exactly one "#," if the "#" does not immediately follow a nonzero digit, or if there are any nonzero digits following the "#." Otherwise, M' accepts if and only if its inter-internal simulation of M accepts.
- When M' reads a digit (any symbol except "#") from the input, it passes that digit to M. The "#" symbol, having been checked as above, is otherwise ignored and is not passed to M.

By the above rules, if M' accepts, then the input must be a Fibonacci number with a "#" inserted following the last nonzero digit. Thus, the input is in the set Q.

Conversely, if the input is in the set Q, then M' will pass the Fibonacci number to M and thus accept.

Therefore, M' accepts the set Q, a contradiction by Lemma 4; hence, the set of Fibonacci numbers is not context-free. □

Acknowledgments

Thanks to Dr. G. E. Bergum for suggesting the investigation of automata recognition of the Fibonacci numbers four years back. Thanks also to Dr. B. Ravikumar for very helpful discussions.

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