ON DETERMINANTS WHOSE ELEMENTS ARE RECURRING SEQUENCES OF ARBITRARY ORDER

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Some years ago, Carlitz [1] and Zeitlin [2] calculated determinants of the form $|w^r_{a+k(i+j)}|$ $(i, j = 0, 1, \ldots, r)$, where $\{w_n\}$ is a second-order recurring sequence. More generally, the aim of this paper is to obtain a closed form for the $s \times s$ determinant

$$(1) \qquad \Delta_w\begin{bmatrix} i_1, & \ldots, & i_r \\ & & \\ j_1, & \ldots, & j_r \end{bmatrix} a = \begin{vmatrix} w_a, & w_{a+j_1}, & \ldots, & w_{a+j_r} \\ w_{a+i_1}, & w_{a+i_1+j_1}, & \ldots, & w_{a+i_1+j_r} \\ \vdots & & & \vdots \\ w_{a+i_r}, & w_{a+i_r+j_1}, & \ldots, & w_{a+i_r+j_r} \end{vmatrix},$$

where $s = r + 1$ and $a, i_1, \ldots, i_r, j_1, \ldots, j_r$ are integers, when $\{w_n\}$ satisfies the recurrence of order s,

$$(2) \qquad w_n = \sum_{k=1}^{s} (-1)^{k-1}\sigma_k w_{n-k}, \quad n \in \mathbf{Z},$$

where $\sigma_1, \sigma_2, \ldots, \sigma_s$ are complex numbers, with $\sigma_s \neq 0$.

We shall often write $\Delta^{i_1, i_2, \ldots, i_r}_{j_1, j_2, \ldots, j_r}$ instead of $\Delta_w\begin{bmatrix} i_1, & \ldots, & i_r \\ j_1, & \ldots, & j_r \end{bmatrix} a$.

We want to obtain an expression of Δ_w in terms of the Fibonacci solution $\{u_n^{(s)}\}$ of (2), whose initial conditions are:

$$(3) \qquad u_0^{(s)} = u_1^{(s)} = \cdots = u_{r-1}^{(s)} = 0; \quad u_r^{(s)} = 1.$$

We define the characteristic number e_w of the sequence $\{w_n\}$ by

$$(4) \qquad e_w = \Delta_w\begin{bmatrix} 1, & 2, & \ldots, & r \\ & & & \\ 1, & 2, & \ldots, & r \end{bmatrix} 0 = |w_{i+j}| \quad (i, j = 0, 1, \ldots, r).$$

Note that, for the Fibonacci sequence $\{u_n^{(s)}\}$, we have, by (3) and (4),

$$e_{u^{(s)}} = (-1)^{\frac{r(r+1)}{2}} = (-1)^{\frac{rs}{2}}.$$

1. A Particular Case

In this section we assume that the characteristic polynomial of (2) admits distinct roots $\alpha_1, \ldots, \alpha_s$, and that α_i/α_j is not a root of unity, for distinct i and j. In that case, there exist complex numbers $C_1, \ldots, C_s$, such that

$$w_n = \sum_{i=1}^{s} C_i \alpha_i^n, \quad n \in \mathbf{Z}.$$

Notice also that

$$\sigma_s = \prod_{i=1}^{s} \alpha_i.$$

The statement of the main result of this section is

Theorem I: $\Delta_w\begin{bmatrix} k, & 2k, & \ldots, & rk \\ k, & 2k, & \ldots, & rk \end{bmatrix}\!\!\left.\begin{matrix}\\ \end{matrix}\right| a \Big] = C_1 \ldots C_s\, \sigma_s^a\, V(\alpha_1^k, \ldots, \alpha_s^k)^2$

$$= e_w\, \sigma_s^a \frac{V(\alpha_1^k, \ldots, \alpha_s^k)^2}{V(\alpha_1, \ldots, \alpha_s)^2},$$

where $V(\alpha_1, \ldots, \alpha_s) = \prod_{i>j} (\alpha_i - \alpha_j)$ is the Vandermonde determinant.

The proof will require the following result.

Lemma I: $e_w = C_1 \ldots C_s V(\alpha_1, \ldots, \alpha_s)^2$.

Proof: From the equality between matrices

$$[w_{i+j}] = [C_{j+1}\alpha_{j+1}^i][\alpha_{i+1}^j] \quad (i, j = 0, 1, \ldots, r),$$

and passing to determinants, we obtain

$$\begin{aligned} e_w &= |C_{j+1}\alpha_{j+1}^i|\,|\alpha_{i+1}^j| \quad (i, j = 0, 1, \ldots, r) \\ &= C_1 \ldots C_s |\alpha_{j+1}^i|^2 \\ &= C_1 \ldots C_s V(\alpha_1, \ldots, \alpha_s)^2. \quad \text{Q.E.D.} \end{aligned}$$

Proof of Theorem I: Let us consider the sequence $\{w_n'\}$, with $w_n' = w_{a+kn}$. Then we have

$$w_n' = \sum_{i=1}^{s} C_i \alpha_i^a (\alpha_i^k)^n, \tag{5}$$

and, since the α_i^k are distinct, $\{w_n'\}$ satisfies a recurrence

$$w_n' = \sum_{m=1}^{s} (-1)^{m-1} \sigma_m' w_{n-m}',$$

with

$$\sigma_m' = \sum_{1 \le i_1 < \ldots < i_m \le s} \alpha_{i_1}^k \ldots \alpha_{i_m}^k.$$

Clearly we have, with the above notations,

$$\Delta_w\left[\begin{matrix} k, & 2k, & \ldots, & rk \\ k, & 2k, & \ldots, & rk \end{matrix}\,\middle|\, a\right] = \Delta_{w'}\left[\begin{matrix} 1, & 2, & \ldots, & r \\ 1, & 2, & \ldots, & r \end{matrix}\,\middle|\, 0\right] = e_{w'}.$$

However, by Lemma I and (5), we have

$$\begin{aligned} e_{w'} &= \left[\prod_{i=1}^{s} C_i \alpha_i^a\right] V(\alpha_1^k, \ldots, \alpha_s^k)^2 \\ &= C_1 \ldots C_s \sigma_s^a V(\alpha_1^k, \ldots, \alpha_s^k)^2 = e_w \sigma_s^a \frac{V(\alpha_1^k, \ldots, \alpha_s^k)^2}{V(\alpha_1, \ldots, \alpha_s)^2}. \end{aligned}$$

Applications:

(i) Put $a = n - rk$ in the formula of Theorem I to get

$$\begin{aligned} \Delta_w\left[\begin{matrix} k, & 2k, & \ldots, & rk \\ k, & 2k, & \ldots, & rk \end{matrix}\,\middle|\, n - rk\right] &= C_1 \ldots C_s \sigma_s^{n-rk} V(\alpha_1^k, \ldots, \alpha_s^k)^2 \\ &= e_w \sigma_s^{n-rk} \frac{V(\alpha_1^k, \ldots, \alpha_s^k)^2}{V(\alpha_1, \ldots, \alpha_s)^2}. \end{aligned} \tag{6}$$

In the case $s = 2$, we obtain

$$\begin{aligned} w_{n-k}w_{n+k} - w_n^2 &= C_1 C_2 \sigma_2^{n-k} (\alpha_1^k - \alpha_2^k)^2 = e_w \sigma_2^{n-k} \frac{(\alpha_2^k - \alpha_1^k)^2}{(\alpha_2 - \alpha_1)^2} \\ &= e_w \sigma_2^{n-k} (u_k^{(2)})^2, \end{aligned}$$

which is the well-known Catalan relation; thus, (6) is a generalization of this result.

(ii) We can also study the sequence $\{w_n^r\}$, where $\{w_n\}$ satisfies the second-order recurrence

$$w_n = pw_{n-1} - qw_{n-2},$$

whence

$$w_n = C_1\alpha_1^n + C_2\alpha_2^n.$$

Assuming that α_1/α_2 is not a root of unity, we get

$$w_n^r = \sum_{i=0}^{r}\binom{r}{i}C_1^iC_2^{r-i}(\alpha_1^i\alpha_2^{r-i})^n, \tag{7}$$

where the $\alpha_1^i\alpha_2^{r-i}$ are distinct. Hence, $\{w_n^r\}$ satisfies a recurrence of type (2), with

$$\sigma_s = \prod_{i=0}^{r}\alpha_1^i\alpha_2^{r-i} = (\alpha_1\alpha_2)^{\frac{rs}{2}} = q^{\frac{rs}{2}}. \tag{8}$$

By application of Theorem I, we obtain a new proof of a known result (see [1], [2]).

Corollary I: $|w^r_{a+k(i+j)}| \qquad (i, j = 0, \ldots, r)$

$$= e_w^{\frac{rs}{2}}\, q^{\frac{ars}{2} + \frac{kr(r^2-1)}{3}} \sum_{i=0}^{r}\binom{r}{i}\sum_{i=1}^{r}(u_{ki}^{(2)})^2.$$

Proof: By Theorem I, (7), and (8), we get

$$\begin{aligned}|w^r_{a+k(i+j)}| &= \Delta_{w^r}\left[\begin{matrix}k, & 2k, & \ldots, & rk\\ k, & 2k, & \ldots, & rk\end{matrix}\middle| a\right] \\ &= \prod_{i=0}^{r}\binom{r}{i}C_1^iC_2^{r-i}\cdot q^{\frac{ars}{2}}\cdot V(\alpha_2^r, \alpha_1\alpha_2^{r-1}, \ldots, \alpha_1^r)^2 \\ &= \prod_{i=0}^{r}\binom{r}{i}\cdot (C_1C_2)^{\frac{rs}{2}}\cdot V(\alpha_2^r, \alpha_1\alpha_2^{r-1}, \ldots, \alpha_1^r)^2,\end{aligned} \tag{9}$$

and it can be shown (see [1], p. 130) that the value of the Vandermonde determinant is

$$(\alpha_1 - \alpha_2)^{\frac{rs}{2}}\, q^{\frac{kr(r^2-1)}{6}} \prod_{i=1}^{r}(u_{ki}^{(2)})^{r-i+1}. \tag{10}$$

The result follows now from (9) and (10) since, by Lemma I,

$$e_w = C_1C_2(\alpha_1 - \alpha_2)^2.$$

2. The General Results

In what follows, we do not make any assumption about the roots of the characteristic equation, and we put again $s = r + 1$. In this section we shall prove the following theorem.

Theorem II: Let $\{w_n\}$ be any solution of the recurrence (2). For all integers $a, i_1, i_2, \ldots, i_r, j_1, j_2, \ldots, j_r$, we have

$$\Delta_w\left[\begin{matrix}i_1, & \ldots, & i_r\\ j_1, & \ldots, & j_r\end{matrix}\middle| a\right] = \sigma_s^a e_w \delta_{i_1,\ldots,i_r}\delta_{j_1,\ldots,j_r}, \tag{11}$$

where $\delta_{i_1, \ldots, i_r}$ is the $r \times r$ determinant

$$\delta_{i_1, \ldots, i_r} = |u^{(s)}_{i_p + q - 1}|, \quad (p, q = 1, 2, \ldots, r).$$

From Theorem II, we get a corollary which can be compared with (6).

Corollary II (Catalan's relation): For all integers n and k, we have

$$\Delta_w\begin{bmatrix} k, & 2k, & \ldots, & rk \\ k, & 2k, & \ldots, & rk \end{bmatrix}\!\!\left. n - rk \right] = \sigma_s^{n-rk} e_w \delta^2_{k, 2k, \ldots, rk}. \tag{12}$$

Proof: Put $a = n - rk$, $j_m = i_m = mk$, $1 \le m \le r$, in the general formula (11).

For example, in the case $s = 2$, (12) becomes

$$w_{n-k}w_{n+k} - w_n^2 = \sigma_2^{n-k}(u_k^{(2)})^2,$$

and, in the case $s = 3$,

$$\begin{vmatrix} w_{n-2k} & w_{n-k} & w_{n+k} \\ w_{n-k} & w_n & w_{n+k} \\ w_n & w_{n+k} & w_{n+2k} \end{vmatrix} = \sigma_3^{n-2k} e_w \begin{vmatrix} u_k^{(3)} & u_{2k}^{(3)} \\ u_{k+1}^{(3)} & u_{2k+1}^{(3)} \end{vmatrix}^2$$

3. Proof of Theorem II

We shall need the following results.

Lemma II:

(i) For all integers $i_1, \ldots, i_r, j_1, \ldots, j_r$,

$$\Delta^{i_1, \ldots, i_r}_{j_1, \ldots, j_r} = \Delta^{j_1, \ldots, j_r}_{i_1, \ldots, i_r}.$$

(ii) For all integers $i_1, \ldots, i_r, j_1, \ldots, j_r$, and all $1 \le p \le r$, we have

$$\Delta^{i_1, \ldots, i_r}_{j_1, \ldots, j_p, \ldots, j_r} = \sum_{k=1}^{s} (-1)^{k-1}\sigma_k \Delta^{i_1, \ldots, i_r}_{j_1, \ldots, j_p - k, \ldots, j_r},$$

and

$$\delta_{i_1, \ldots, i_p, \ldots, i_r} = \sum_{k=1}^{s} (-1)^{k-1}\sigma_k \delta_{i_1, \ldots, i_p - k, \ldots, i_r}.$$

(iii) If τ is a permutation of $\{1, 2, \ldots, r\}$ of sign $\varepsilon(\tau)$, then for all integers $i_1, \ldots, i_r, j_1, \ldots, j_r$,

$$\Delta^{i_1, \ldots, i_r}_{j_{\tau(1)}, \ldots, j_{\tau(r)}} = \varepsilon(\tau)\Delta^{i_1, \ldots, i_r}_{j_1, \ldots, j_r},$$

and

$$\delta_{j_{\tau(1)}, \ldots, j_{\tau(r)}} = \varepsilon(\tau)\delta_{j_1, \ldots, j_r}.$$

(iv) If $j_k = j_\ell$ for distinct k and ℓ or if there exists k such that $j_k = 0$, then

$$\Delta^{i_1, \ldots, i_r}_{j_1, \ldots, j_r} = \delta_{j_1, \ldots, j_r} = 0.$$

Proof: This is an immediate consequence of the properties of determinants.

Lemma III: Let us consider two sequences $\{X_n\}$ and $\{Y_n\}$, with $n = (n_1, \ldots, n_t) \in \mathbb{Z}^t$, such that, for all $n \in \mathbb{Z}^t$, and all $1 \le p \le t$,

$$X_{n_1, \ldots, n_p, \ldots, n_t} = \sum_{k=1}^{s} (-1)^{k-1}\sigma_k X_{n_1, \ldots, n_p - k, \ldots, n_t}, \tag{13}$$

and

$$Y_{n_1, \ldots, n_p, \ldots, n_t} = \sum_{k=1}^{s} (-1)^{k-1}\sigma_k Y_{n_1, \ldots, n_p - k, \ldots, n_t}. \tag{14}$$

If $X_n = Y_n$ holds for all n belonging to

(15) $$C_t = \{n \in \mathbf{Z}^t,\ 0 \le n_p \le r,\ 1 \le p \le t\},$$

then

(16) $$X_n = Y_n \text{ holds for all } n \in \mathbf{Z}^t.$$

Proof: By induction on t. The statement is well known for $t = 1$. Let us suppose that (16) holds up to a certain $t \ge 1$. For the inductive step $t \to t + 1$, fix an integer m and consider the sequences $\{x_n^{(m)}\}$ and $\{y_n^{(m)}\}$, with $n = (n_1, \ldots, n_t)$ defined by

$$x_n^{(m)} = X_{n_1, \ldots, n_t, m} \quad \text{and} \quad y_n^{(m)} = Y_{n_1, \ldots, n_t, m}.$$

By definition, $x_n^{(m)} = y_n^{(m)}$ holds for all $n \in C_t$ and all $0 \le m \le r$, and by the induction hypothesis,

$$x_n^{(m)} = y_n^{(m)} \text{ for } n \in \mathbf{Z}^t \text{ and } 0 \le m \le r.$$

Now, fix $n \in \mathbf{Z}^t$ and consider the sequences x'_m and y'_m, defined by

$$x'_m = X_{n_1, \ldots, n_t, m} \quad \text{and} \quad y'_m = Y_{n_1, \ldots, n_t, m}.$$

We have $x'_m = y'_m$ for $0 \le m \le r$, and the same equality holds for all integers m, since by (13) $\{x'_m\}$ and $\{y'_m\}$ satisfy a recurrence relation of order s. This concludes the proof of Lemma 3.

Proof of Theorem 2:

<u>Step 1</u>: We prove that, for all integers $i_1, \ldots, i_r, j_1, \ldots, j_r$,

(17) $$\Delta_{j_1, \ldots, j_r}^{i_1, \ldots, i_r} = \Delta_{1, 2, \ldots, r}^{i_1, \ldots, i_r} (-1)^{\frac{r(r-1)}{2}} \delta_{j_1, \ldots, j_r}.$$

Let us fix $i_1, \ldots, i_r$. By Lemma 2(ii) and Lemma 3, it suffices to show that (17) holds for $j_1, \ldots, j_r$ belonging to the set

$$C_r = \{(j_1, \ldots, j_r) \in \mathbf{Z}^r,\ 0 \le j \le r,\ 1 \le p \le r\}.$$

If one of the conditions of Lemma 2(iv) is satisfied, then (17) clearly holds. Therefore, we have only to consider the case where $(j_1, \ldots, j_r)$ is a permutation of $(1, 2, \ldots, r)$. By a direct calculation,

$$\delta_{1, \ldots, r} = (-1)^{\frac{r(r-1)}{2}},$$

whence (17) holds for $(j_1, \ldots, j_r) = (1, 2, \ldots, r)$, and by Lemma 2(iii), the equality holds for every permutation of $(1, 2, \ldots, r)$.

<u>Step 2</u>: By Lemma 2(i) and Step 1, the following statement holds:

$$\Delta_{1, 2, \ldots, r}^{i_1, \ldots, i_r} = \Delta_{i_1, \ldots, i_r}^{1, 2, \ldots, r} = \Delta_{1, 2, \ldots, r}^{1, 2, \ldots, r} (-1)^{\frac{r(r-1)}{2}} \delta_{i_1, \ldots, i_r}.$$

Hence, (17) becomes

(18) $$\Delta_{j_1, \ldots, j_r}^{i_1, \ldots, i_r} = \Delta_{1, 2, \ldots, r}^{1, 2, \ldots, r} \delta_{i_1, \ldots, i_r} \delta_{j_1, \ldots, j_r}.$$

Now, it is known (see [3], p. 99) that

$$\Delta_{1, 2, \ldots, r}^{1, 2, \ldots, r} = \delta_s^a e_w.$$

By this and (18), the proof is complete.

For a second-order recurring sequence, (11) becomes

$$w_a w_{a+i+j} - w_{a+i} w_{a+j} = \sigma_2^a e_w u_i^{(2)} u_j^{(2)}.$$

When giving particular values to a, i, and j, one can deduce from this some well-known identities.

References

1. L. Carlitz. "Some Determinants Containing Powers of Fibonacci Numbers." *Fibonacci Quarterly* 4.2 (1966):129-34.
2. D. Zeitlin. "On Determinants Whose Elements Are Products of Recursive Sequences." *Fibonacci Quarterly* 8.4 (1970):350-59.
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Announcement

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The FIFTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at The University of St. Andrews, St. Andrews, Scotland from July 20 to July 24, 1992. This Conference is sponsored jointly by the Fibonacci Association and The University of St. Andrews.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts are to be submitted by March 15, 1992. Manuscripts are due by May 30, 1992. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of *The Fibonacci Quarterly* to:

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