A NEW FORMULA FOR LUCAS NUMBERS

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Introduction

The Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$ are well-known to the readers of this Journal. Several closed form formulas exist for Fibonacci and Lucas numbers, namely:

(1) $$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$ (2) $$L_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{1}{2}(1 + 5^{\frac{1}{2}})$, $\beta = \frac{1}{2}(1 - 5^{\frac{1}{2}})$.

(3) $$F_n = \frac{1}{2^{n-1}} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} 5^k,$$ (4) $$L_n = \frac{1}{2^{n-1}} \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} 5^k,$$

(5) $$F_{n+1} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k}$$ (6) $$L_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k} \binom{n-k}{k}.$$

George E. Andrews, [1] and [2], derived an additional explicit formula for the Fibonacci numbers, which can be written as

(7) $$F_n = \sum_{k=-\left[\frac{n+1}{5}\right]}^{\left[\frac{n}{5}\right]} (-1)^k \binom{n}{[\frac{1}{2}(n - 5k)]}.$$

In [1], Andrews proved (7) by using a relation between the Fibonacci numbers and the primitive fifth roots of unity, namely:

$$\alpha = -2\cos(4\pi/5), \quad \beta = -2\cos(2\pi/5).$$

In [2], Andrews obtained (7) as a consequence of a polynomial identity. In this note, following Andrews, we derive a corresponding explicit formula for the Lucas numbers which is

(8) $$L_n = \sum_{k=-\left[\frac{n+1}{5}\right]}^{\left[\frac{n}{5}\right]} (-1)^k \frac{n + [\frac{1}{2}(n - 5k)]}{n} \binom{n}{[\frac{1}{2}(n - 5k)]}.$$

Preliminaries

(9) $$\sum_{j=0}^{\left[\frac{n}{2}\right]} x^{j^2+j} \prod_{k=1}^{j} \frac{x^{n+1-j-k} - 1}{x^k - 1} = \sum (-1)^t x^{\frac{1}{2}t(5t-3)} \prod_{k=1}^{\left[\frac{n+3-5t}{2}\right]} \frac{x^{n+2-k} - 1}{x^k - 1}.$$

(10) $$F_{n+1} = \sum (-1)^k \binom{n+1}{[\frac{1}{2}(n + 1 - 5k)] + 1}.$$

(11) $$\binom{n}{k} = \binom{n}{n-k}.$$

(12) $m = \left[\frac{m + r}{2}\right] + \left[\frac{m + 1 - r}{2}\right]$ for all m, r.

(13) $\binom{m - 1}{r - 1} = \frac{r}{m}\binom{m}{r}$ if $1 \le r \le m$.

(14) $L_n = F_{n+1} + F_{n-1}$.

Remarks: Equation (9) is the Theorem from [2] with $\alpha = -1$. Equation (10) is obtained by taking the limit as x approaches 1 in (9) and then applying (5). Equations (11) through (14) are elementary.

Proof of (8): Equation (10) implies that

(15) $F_{n-1} = \sum (-1)^k \binom{n - 1}{[\frac{1}{2}(n - 1 - 5k)] + 1}$.

Replacing k by $-k$ in (15), we get

(16) $F_{n-1} = \sum (-1)^k \binom{n - 1}{[\frac{1}{2}(n - 1 + 5k)] + 1}$.

which implies, by using (11), that

(17) $F_{n-1} = \sum (-1)^k \binom{n - 1}{n - 2 - [\frac{1}{2}(n - 1 + 5k)]}$.

If we now use equation (12), we see that

(18) $F_{n-1} = \sum (-1)^k \binom{n - 1}{[\frac{1}{2}(n - 5k)]}$.

Applying (13) to equation (18), we obtain

(19) $F_{n-1} = \sum (-1)^k \frac{[\frac{1}{2}(n - 5k)]}{n}\binom{n}{[\frac{1}{2}(n - 5k)]}$.

Equation (19) together with equations (7) and (14) yields

(20) $L_n = \sum (-1)^k \frac{n + [\frac{1}{2}(n - 5k)]}{n}\binom{n}{[\frac{1}{2}(n - 5k)]}$,

which is the same as (8) and the proof is complete. (The limits of summation in (8) are determined by the criterion that $0 \le [\frac{1}{2}(n - 5k)] \le n$.)

Concluding Remarks

The reader who consults [1] should take note that (i) Andrews' middle initial is erroneously given as H.; (ii) on pages 113 and 117, the name "Einstein" should be "Eisenstein." Both errors were made without consulting Andrews and were not in his original manuscript.

Acknowledgment

I wish to thank the referee for his suggestions, which led to a simpler proof of (8).

References

1. George E. Andrews. "Some Formulae for the Fibonacci Sequence with Generalizations." *Fibonacci Quarterly* 7.2 (1969):113-30.
2. George E. Andrews. "A Polynomial Identity which Implies the Rogers-Ramanujan Identities." *Scripta Math.* 28 (1970):297-305.

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