## ADVANCED PROBLEMS AND SOLUTIONS

Edited by VERNER E. HOGGATT, JR.
San Jose State College, San Jose, California

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-52 *Proposed by Brother U. Alfred, St. Mary's College, California*

Prove that the value of the determinant:

$$\begin{vmatrix} u_n^2 & u_{n+2}^2 & u_{n+4}^2 \\ u_{n+2}^2 & u_{n+4}^2 & u_{n+6}^2 \\ u_{n+4}^2 & u_{n+6}^2 & u_{n+8}^2 \end{vmatrix}$$

is $18\,(-1)^{n+1}$.

H-53 *Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, California and S.L. Basin, Sylvania Electronics Systems, Mt. View, California*

The Lucas sequence, $L_1 = 1$, $L_2 = 3$; $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$, is incomplete (see V. E. Hoggatt, Jr. and C. King, Problem E-1424 American Mathematical Monthly Vol. 67, No. 6, June-July 1960 p. 593) since every integer n, is not the sum of distinct Lucas numbers. OBSERVE THAT 2, 6, 9, 13, 17, ... cannot be so represented. Let $M(n)$ be the number of positive integers less than n which cannot be so represented. Show

$$M(L_n) = F_{n-1} .$$

Find, if possible, a closed form solution for $M(n)$.

H-54 *Proposed by Douglas Lind, Falls Church, Va.*

If $F_n$ is the nth Fibonacci number, show that

$$\phi(F_n) \equiv 0 \pmod 4), \qquad n > 4$$

where $\phi(n)$ is Euler's function.

H-55 *Proposed by Raymond Whitney, Lock Haven State College, Lock Haven, Penn.*

Let F(n) and L(n) denote the nth Fibonacci and nth Lucas numbers, respectively.

Given U(n) = F(F(n)), V(n) = F(L(n)), W(n) = L(L(n)) and X(n) = $L(F_n)$, find recurrence relations for the sequences U(n), V(n), W(n) and X(n).

H-56 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Show

$$\sum_{n=1}^{\infty} \frac{F_k^n}{F_n F_{n+2} \cdots F_{n+k} F_{n+k+1}} = \frac{(F_k/F_{k+1})}{\prod_{i=1}^{k+1} F_i}, \quad k \geq 1 .$$

H-57 *Proposed by George Ledin, Jr., San Francisco, California*

If $F_n$ is the nth Fibonacci number, define

$$G_n = \left( \sum_{k=1}^{n} k F_k \right) \Bigg/ \left( \sum_{k=1}^{n} F_k \right)$$

and show

$$\text{(i)} \quad \lim_{n \to \infty} (G_{n+1} - G_n) = 1$$

$$\text{(ii)} \quad \lim_{n \to \infty} (G_{n+1}/G_n) = 1 .$$

Generalize.

H-58 *Proposed by John L. Brown, Jr., Ordnance Research Laboratory, The Penn. State University, State College, Penn.*

Evaluate, as a function of n and k, the sum

$$\sum_{i_1+i_2+\cdots+i_{k+1}=n} F_{2i_1+2} F_{2i_2+2} \cdots F_{2i_k+2} F_{2i_{k+1}+2} ,$$

where $i_1, i_2, i_3, \ldots, i_{k+1}$ constitute an ordered set of indices which take on the values of all permutations of all sets of k+1 nonnegative integers whose sum is n.

## REPROPOSED CHALLENGE

H-22 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

$$\text{If } P(x) = \prod_{i=1}^{\infty} (1 + x^{F_i}) = \sum_{n=0}^{\infty} R(n)\, x^n ,$$

then show

(i) $$R(F_{2n} - 1) = n$$

(ii) $$R(N) > n \text{ if } N > F_{2n} - 1 \quad .$$

INVERSION OF FIBONACCI POLYNOMIALS

P-3 *Proposed by Paul F. Byrd, San Jose State College, San Jose, California* in "Expansion of Analytic Functions in Polynomials Associated with Fibonacci Numbers," Vol. I, No. 1, Feb. 1963, pp. 16-29.

Verify the reciprocal relationship

$$x^n = (\tfrac{1}{2})^n \sum_{r=0}^{[n/2]} (-1)^r \binom{n}{r} \frac{n-2r+1}{n-r+1} \gamma_{n+1-2r}(x), \quad (n \geq 0)$$

where

$$\gamma_{k+1}(x) = \sum_{m=0}^{[k/2]} \binom{k-m}{m} (2x)^{k-2m} \quad (k \geq 0)$$

*Solution by Gary McDonald, St. Mary's College, Winona, Minnesota*

Verification by induction: (Equation numbers refer to P. F. Byrd's article,

For $n = 0$, we have $1 = \gamma_1(x)$ which agrees with (2.2). Assuming (1) true for $n = k$, we can write

$$x^{k+1} = \frac{1}{2^{k+1}} \sum_{r=0}^{[k/2]} (-1)^r \binom{k}{r} \frac{k-2r+1}{k-r+1} (2x) \gamma_{k+1-2r}(x) \quad .$$

Recalling (2.1), we have

$$x^{k+1} = \frac{1}{2^{k+1}} \left[ \sum_{r=0}^{[k/2]} (-1)^r \binom{k}{r} \frac{k-2r+1}{k-r+1} \gamma_{k+2-2r}(x) - \sum_{r=0}^{[k/2]} (-1)^r \binom{k}{r} \frac{k-2r+1}{k-r+1} \gamma_{k-2r}(x) \right]$$

$$= \frac{1}{2^{k+1}} \left[ \gamma_{k+2}(x) + \sum_{r=1}^{[k/2]} (-1)^r \binom{k}{r} \frac{k-2r+1}{k-r+1} \gamma_{k+2-2r}(x) - \right.$$

$$\left. - \sum_{r=0}^{[k/2]-1} (-1)^r \binom{k}{r} \frac{k-2r+1}{k-r+1} \gamma_{k-2r}(x) + C \right]$$

where

$$C = (-1)^{[k/2]+1}\binom{k}{[k/2]}\frac{k-2[k/2]+1}{k-[k/2]+1}\,\gamma_{k-2[k/2]}(x)\ .$$

Letting $j = r + 1$ in the second $\Sigma$ for $x^{k+1}$,

$$x^{k+1} = \frac{1}{2^{k+1}}\Bigg[\gamma_{k+2}(x) + \sum_{r=1}^{[k/2]}(-1)^r\binom{k}{r}\frac{k-2r+1}{k-r+1}\,\gamma_{k+2-2r}(x) + \sum_{j=1}^{[k/2]}(-1)^j\binom{k}{j-1}\frac{k+3-2j}{k+2-j}\,\gamma_{k+2-2j}(x) + C\Bigg]\ ,$$

or combining coefficients of $\gamma_i(x)$

$$x^{k+1} = \frac{1}{2^{k+1}}\Bigg\{\gamma_{k+2}(x) + \sum_{r=1}^{[k/2]}(-1)^r\left[\binom{k}{r}\frac{k-2r+1}{k-r+1} + \binom{k}{r-1}\frac{k+3-2r}{k+2-r}\right]\gamma_{k+2-2r}(x) + C\Bigg\} \tag{2}$$

We can reduce the quantity in brackets as follows:

$$\binom{k}{r}\frac{k-2r+1}{k-r+1} + \binom{k}{r-1}\frac{k+3-2r}{k+2-r} = \left[\binom{k}{r}\frac{(k-2r+1)(k+2-r)}{k-r+1} + \binom{k}{r-1}(k+3-2r)\right]\left(\frac{1}{k+2-r}\right)$$

$$= \left[\frac{k!}{(k-r+1)!\,r!}(k-2r+1)(k+2-r) + \frac{k!\,(k+3-2r)}{(k-r+1)!\,(r-1)!}\right]\left(\frac{1}{k+2-r}\right)$$

$$= \left[\frac{(k-2r+1)(k+2-r)+r(k+3-2r)}{k+1}\right]\frac{(k+1)!}{(k-r+1)!\,r!\,(k+2-r)}$$

$$= \left[\frac{k^2+3k-2rk-2r+2}{(k+1)}\right]\frac{(k+1)!}{(k-r+1)!\,r!\,(k+2-r)}$$

$$= \frac{k+2-2r}{k+2-r}\binom{k+1}{r}\ .$$

Therefore from (2),

$$x^{k+1} = \frac{1}{2^{k+1}}\left[\gamma_{k+2}(x) + \sum_{r=1}^{[k/2]}(-1)^r\binom{k+1}{r}\frac{k+2-2r}{k+2-r}\,\gamma_{k+2-2r}(x) + C\right]. \tag{3}$$

Note that:

a) $$\gamma_{k+2}(x) = (-1)^0\binom{k+1}{0}\frac{k+2-0}{k+2-0}\,\gamma_{k+2-0}(x)\ .$$

b) When $k$ is even $C = 0$, and $\left[\frac{k}{2}\right] = \left[\frac{k+1}{2}\right]$.

When $k$ is odd, then $\left[\frac{k}{2}\right] = \frac{k-1}{2}$ and

$$C = (-1)^{\left[\frac{k+1}{2}\right]}\binom{k}{\frac{k-1}{2}} \frac{2}{\frac{3+k}{2}} \gamma_1(x)$$

$$= (-1)^{\left[\frac{k+1}{2}\right]} \frac{k!}{(\frac{k-1}{2})!\,(\frac{k+1}{2})!} \frac{4}{3+k} \gamma_1(x)$$

If we let $r = \left[\frac{k+1}{2}\right]$ in the $\Sigma$ of equation (3), we have

$$(-1)^{\left[\frac{k+1}{2}\right]} \frac{(k+1)!}{(\frac{k+1}{2})!\,(\frac{k+1}{2})!} \frac{2}{k+3} \gamma_1(x) = (-1)^{\left[\frac{k+1}{2}\right]} \frac{(k+1)k!}{\frac{2(k+1)}{2}(\frac{k-1}{2})!\,(\frac{k+1}{2})!} \frac{4}{k+3} \gamma_1(x)$$

$$= (-1)^{\left[\frac{k+1}{2}\right]} \frac{k!}{(\frac{k-1}{2})!\,(\frac{k+1}{2})!} \frac{4}{k+3} \gamma_1(x)$$

$$= C, \quad k \text{ odd.}$$

Therefore, we may combine $\gamma_{k+2}(x)$ and $C$ into the $\Sigma$ in (3) and write

$$x^{k+1} = \frac{1}{2^{k+1}} \sum_{r=0}^{\left[\frac{k+1}{2}\right]} (-1)^r \binom{k+1}{r} \frac{k+2-2r}{k+2-r} \gamma_{k+2-2r}(x) ,$$

from which we conclude

$$x^n = \frac{1}{2^n} \sum_{r=0}^{[n/2]} (-1)^r \binom{n}{r} \frac{n-2r+1}{n-r+1} \gamma_{n+1-2r}(x), \qquad n \geq 0 .$$

DEFERRED ANSWER

H-34 *Proposed by P.F. Byrd, San Jose State College*

Derive the series expansions

$$J_{2k}(\alpha) = I_k^2(\alpha) + \sum_{m=1}^{\infty} (-1)^{m+k} I_{m+k}(\alpha) I_{m-k}(\alpha) L_{2m}$$

($k = 0, 1, 2, 3, \ldots$) for the Bessel functions $J_{2k}$ of all even orders, where $L_n$ are Lucas numbers and $I_n$ are modified Bessel functions.

The solution will appear in a fine paper by the proposer to appear later in the Quarterly.

## FIBONACCI AND MAGIC SQUARES

H-35 *Proposed by Walter W. Horner, Pittsburgh, Pa.*

Select any nine consecutive terms of the Fibonacci sequence and form the magic square

| | | |
|---|---|---|
| $u_8$ | $u_1$ | $u_6$ |
| $u_3$ | $u_5$ | $u_7$ |
| $u_4$ | $u_9$ | $u_2$ |

show

$$u_8u_1u_6 + u_3u_5u_7 + u_4u_9u_2 = u_8u_3u_4 + u_1u_5u_9 + u_6u_7u_2 \quad .$$

Generalize.

*Solution by Maxey Brooke, Sweeny, Texas and F.D. Parker, SUNY, Buffalo, N.Y.*

If $U_n$ satisfies the general second order difference equation, then

$$\begin{vmatrix} U_1 & U_2 & U_3 \\ U_4 & U_5 & U_6 \\ U_7 & U_8 & U_9 \end{vmatrix} = 0$$

since $U_{n+2} = \alpha U_{n+1} + \beta U_n$ with $U_1$ and $U_2$ arbitrary. The expansion of this determinant yields products whose subscripts add up to the requisite 15 and yields the equality asked for in the problem.

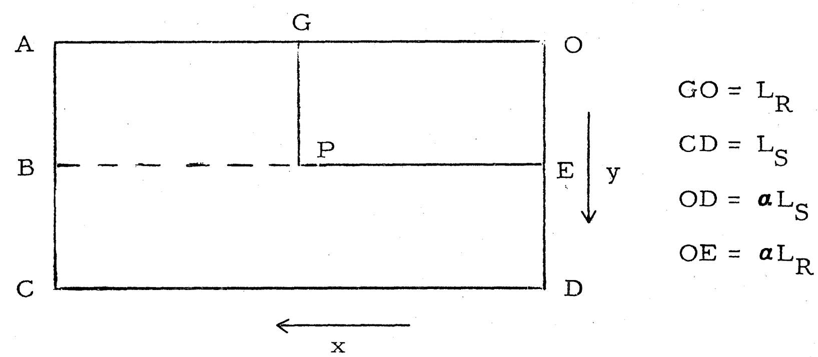
*Also solved by the proposer.*

## GOLDEN SECTION IN CENTROIDS

H-36 *Proposed by J.D.E. Konhauser, State College, Pa.*

Consider a rectangle R. From the upper right corner of R remove a rectangle S (similar to R and with sides parallel to the sides of R. Determine the linear ratio $K = L_R/L_S$ if the centroid of the remaining L shaped region is where the lower left corner of the removed rectangle was.

*Solution by John Wessner, Melbourne High School, Melbourne, Florida*




The centroid of AGPB is at $\frac{1}{2}\{L_S + L_R\}, \frac{1}{2}\alpha L_R$ and has weight $\alpha L_R\{L_S - L_R\}$. Similarly the centroid of BEDC is at $\frac{1}{2}L_S$, $\frac{1}{2}\alpha\{L_S + L_R\}$ and has weight $\alpha L_S\{L_S - L_R\}$. The centroid of this remainder must be at P and have x-coordinate

$$L_R = \frac{\frac{1}{2}L_S^2\alpha\{L_S - L_R\} + \frac{1}{2}\alpha\{L_S + L_R\}\{L_S - L_R\}L_R}{\alpha\{L_S^2 - L_R^2\}}$$

or upon expanding

$$2L_R\{L_S^2 - L_R^2\} = L_S^3 - L_R^3 \quad .$$

Division by $L_R^3$ gives

$$2\{K^2 - 1\} = K^3 - 1 \quad .$$

After removing the obvious root +1 we have $K^2 - K - 1 = 0$ which has as its positive root $\gamma = (1 + \sqrt{5})/2$.

Editorial Comment: The above property is shared by many geometric figures including the ellipse. A short paper later will show this.

*Also solved by David Sowers and the proposer.*

## A FASCINATING RECURRENCE

H-37 *Proposed by H.W. Gould, West Virginia University, Morgantown, West. Va.*

Find a triangle with sides n+1, n, n-1 having integral area. The first two examples appear to be 3, 4, 5 with area 6; and 13, 14, 15 with area 84.

The proposer's paper comprehensively discussing this problem will soon appear in the Quarterly.

NO SOLUTIONS RECEIVED

H-38 *Proposed by R.G. Buschman, SUNY, Buffalo, N.Y.*

(See Fibonacci Numbers, Chebyshev Polynomials, Generalizations and Difference Equations Vol. 1, No. 4, Dec. 1963, pp. 1-7.)

Show $$(u_{n+r} + (-b)^r u_{n-r})/u_n = \lambda_r .$$

CORRECTED

H-40 *Proposed by Walter Blumberg, New Hyde Park, L.I., N.Y.*

Let U, V, A and B be integers, subject to the following conditions (i) $U > 1$, (ii) $(U, 3) = 1$; (iii) $(A, V) = 1$;

$$\text{(iv)} \quad V = \sqrt{(U^2 - 1)/5} \quad .$$

Show $A^2U+BV$ is not a square.

CONVOLUTIONS AND OPTICAL 2-STACK

H-39 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Solve the difference equation in closed form

$$C_{n+2} = C_{n+1} + C_n + F_{n+2} \quad ,$$

where $C_1 = 1$, $C_2 = 2$, and $F_n$ is the nth Fibonacci number. Give two separate characterizations of these numbers.

*Solution by L. Carlitz, Duke University, Durham, N.C.*

Since $C_2 = C_1 + C_0 + F_2$ we have also $C_0 = 0$. If we put

$$C(t) = \sum_0^\infty C_n t^n \quad ,$$

then it follows from

$$C_{n+2} = C_{n+1} + C_n + F_{n+2}$$

that

$$(1-t-t^2)C(t) = \sum_0^\infty F_n t^n = \frac{t}{1-t-t^2} \quad .$$

Thus

$$C(t) = \frac{t}{(1-t-t^2)^2} \quad . \qquad (1)$$

Expanding we get

$$C(t) = t \sum_{r=0}^{\infty} (r+1)(t+t^2)^r$$

$$= \sum_{r=0}^{\infty} (r+1)t^{r+1} \sum_{s=0}^{r} \binom{r}{s} t^s$$

$$= \sum_{n=0}^{\infty} t^{n+1} \sum_{r=0}^{n} (r+1)\binom{r}{n-r} \quad ,$$

so that

$$C_{n+1} = \sum_{r=0}^{n} (r+1)\binom{r}{n-r} = \sum_{2r \le n} (n-r+1)\binom{n-r}{r} \quad .$$

Another explicit expression that follows from (1) is

$$C_{n-1} = \sum_{r=1}^{n-1} F_r F_{n-r} \quad .$$

Next is we differentiate

$$\frac{t}{1-t-t^2} = \sum_{0}^{\infty} F_n t^n$$

we get

$$\frac{1+t^2}{(1-t-t^2)^2} = \sum_{0}^{\infty} (n+1)F_{n+1} t^n \quad ,$$

which yields

$$C_n + C_{n-2} = (n+1)F_{n+1} \quad .$$

A consequence of this is

$$C_n = \sum_{2k \le n} (-1)^k (n-2k+1)F_{n-2k+1} \quad .$$

Finally consider the number

$$C'_n = AnF_n + BnL_n \quad .$$

We find that

$$C'_{n+2} - C'_{n+1} - C'_n = A(F_{n+2}+F_n) + B(L_{n+2}+L_n) \quad .$$

Since

$$L_{n+2} + L_n = 5(F_{n+2}-F_n)$$

we get

$$C'_{n+2} - C'_{n+1} - C'_n = (A+5B)F_{n+2} + (A-5B)F_n \quad .$$

Hence for $A = 6$, $B = 1/10$ it follows that

$$C'_{n+2} - C'_{n+1} - C'_n = F_{n+2} \quad .$$

Clearly

$$C_n = n/2\ F_n + n/10\ L_n + aF_n + bL_n \quad .$$

Taking $n = 0$ we get $b = 0$. For $n = 1$ we get $a = 2/3$. Therefore we have

$$C_n = n/2\ F_n + n/10\ L_n + 2/5\ F_n = \frac{n\ L_{n+1} + 2F_n}{5} \quad .$$

*Also solved by Ronald Weimshenk, John L. Brown, Jr., Donald Knuth, H.H. Ferns and the proposer.*

Editorial Note: Another characterization, besides the convolution

$$C_{n+1} = \sum_{r=1}^{n+1} F_r F_{n-r} = \frac{(n+1)L_{n+2} + 2F_{n+1}}{5} \quad ,$$

is the number of crossings of the interface, in the optical stack in problem B-6, Dec. 1963, p. 75, for all rays which are reflected n-times. If $f_0(x) = 0$, $f_1(x) = 1$, and $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$, the Fibonacci polynomials, then

$$f_n(1) = F_n \quad \text{and} \quad f'_n(1) = C_{n-1} \quad .$$

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MATH MORALS

Brother U. Alfred

A tutor who tutored two rabbits,
Was intent on reforming their habits.
Said the two to the tutor,
"There are rabbits much cuter,
But non-Fibonacci, dagnabits."*

*The author has just taken out poetic license $\#F_{97}$ according to one clause of which it is permissible to corrupt corrupted words.

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