FIBONACCI FANTASY: THE SQUARE ROOT OF THE Q MATRIX

MARJORIE BICKNELL
Adrian Wilcox High School, Santa Clara, California

The matrix

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

has many well known fascinating properties, one being that

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

where F_n is the nth Fibonacci number. The Q matrix also has a Fibonacci square root, which can be exhibited after making a simple definition.

We extend the relationships

$$L_k = \alpha^k + \beta^k$$

$$F_k = (\alpha^k - \beta^k)/(\alpha - \beta), \quad \alpha = (1 + \sqrt{5})/2, \quad \beta = (1 - \sqrt{5})/2,$$

to allow k to equal any integral multiple of one-half. Considering odd multiples of one-half for a moment, it is easy algebraically to obtain

$$F^2_{(2n+1)/2} = \left[L_{2n+1} + 2i(-1)^{n+1}\right]/5$$

and

$$L^2_{(2n+1)/2} = L_{2n+1} + 2i(-1)^n, \quad i = \sqrt{-1},$$

directly from the extended definition. So then, all Fibonacci or Lucas numbers whose subscripts are odd multiples of one-half are complex. Also, combining the two equations directly above yields

$$L^2_{(2n+1)/2} - 5F^2_{(2n+1)/2} = 4i(-1)^n.$$

Returning to the Q matrix, a square root of Q is given by [1]

$$Q^{1/2} = \begin{bmatrix} F_{3/2} & F_{1/2} \\ F_{1/2} & F_{-1/2} \end{bmatrix} ,$$

for, by applying the extended definition and simplifying,

$$F_{3/2}^2 + F_{1/2}^2 = 1, \quad F_{1/2}^2 + F_{-1/2}^2 = 0 ,$$

and

$$F_{3/2}F_{1/2} + F_{1/2}F_{-1/2} = 1 .$$

As suggested by the second of these equalities, we can write

$$F_{-1/2} = iF_{1/2} .$$

By taking the determinant of the square root of Q,

$$i = F_{3/2}F_{-1/2} - F_{1/2}^2 .$$

Also, that

$$Q^{n/2} = \begin{bmatrix} F_{(n+2)/2} & F_{n/2} \\ F_{n/2} & F_{(n-2)/2} \end{bmatrix}$$

can be established by induction, using the extended definition and algebraic manipulation. By equating corresponding elements of equal matrices, from $(Q^{n/2})^2 = Q^n$, we obtain

$$F_{(n+2)/2}^2 + F_{n/2}^2 = F_{n+1}$$

and

$$F_{(n+2)/2}F_{n/2} + F_{n/2}F_{(n-2)/2} = F_n .$$

Taking the determinant of $Q^{n/2}$ yields

$$F_{(n+2)/2}F_{(n-2)/2} - F_{n/2}^2 = (-1)^{n/2} = i^n .$$

The reader can easily establish that

$$F_{(2n+3)/2} = F_{3/2}F_{n+1} + F_nF_{1/2} ,$$

$$F_{(2n+3)/2} = F_{(n+3)/2}F_{(n+2)/2} + F_{(n+1)/2}F_{n/2} ,$$

$$F_{2n+1} = F_{(2n+1)/2}L_{(2n+1)/2} ,$$

$$F_{(2n+1)/2} = F_{(n+3)/2}F_{n/2} + F_{(n+1)/2}F_{(n-2)/2} .$$

Let us pursue a more general result. It can be established by induction that

$$Q^{p/r} = \begin{bmatrix} F_{(p+r)/r} & F_{p/r} \\ F_{p/r} & F_{(p-r)/r} \end{bmatrix}, \quad r \neq 0 ,$$

if we further extend the definition of Fibonacci numbers to include subscripts which are rational numbers. Taking the determinant yields

$$(-1)^{p/r} = F_{(p+r)/r}F_{(p-r)/r} - F^2_{p/r} .$$

As an example, since

$$Q^{p/r}\, Q^{r/p} = Q^{(p^2+r^2)/rp},$$

consideration of the elements of these matrices leads to

$$F_{(p^2+r^2+rp)/rp} = F_{(p+r)/r}F_{(r+p)/p} + F_{p/r}F_{r/p} ,$$

which is a general case of the familiar identity

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n .$$

In general, it seems that identities which hold for integral subscripts also hold for our specialized rational subscripts. What if the Fibonacci subscript were a complex number? J. C. Amson [2] has answered this question while demonstrating an analogy to the familiar circular and hyperbolic functions.

Amson defined modified Lucas functions as

$$\text{luc } z = (w^z - \bar{w}^z)/2\Delta, \qquad \text{coluc } z = (w^z + \bar{w}^z)/2,$$

where z is a complex number and w and $\bar{w}$ are the roots of the quadratic equation $x^2 = Px - Q$, with discriminant $\Delta^2 = P^2 - 4Q$. (Notice that luc $z = (F_z)/2$, coluc $z = (L_z)/2$ when $P = 1$, $Q = -1$.) Algebraically, we see that, among other identities,

$$Q^z \text{luc}(-z) = -\text{ luc } z$$

$$Q^z \text{coluc}(-z) = \text{coluc } z$$

$$\text{luc } 0 = 0, \text{ coluc } 0 = 1$$

$$\text{luc } 2z = 2 \text{ luc } z \text{ coluc } z$$

$$\text{luc}(z_1 + z_2) = \text{luc } z_1 \text{ coluc } z_2 + \text{coluc } z_1 \text{ luc } z_2$$

$$Q^z \text{ luc}(z_1 - z_2) = \text{luc } z_1 \text{ coluc } z_2 - \text{coluc } z_1 \text{ luc } z_2$$

$$\text{coluc}(z_1 + z_2) = \text{coluc } z_1 \text{ coluc } z_2 + \Delta^2 \text{luc } z_1 \text{ luc } z_2$$

$$Q^z \text{ coluc}(z_1 - z_2) = \text{coluc } z_1 \text{ coluc } z_2 - \Delta^2 \text{luc } z_1 \text{ luc } z_2$$

$$\text{coluc}^2 z + \Delta^2 \text{luc}^2 z = \text{coluc } 2z$$

$$\text{coluc}^2 z - \Delta^2 \text{luc}^2 z = Q^z$$

$$(\text{coluc } z + \Delta \text{ luc } z)^n = \text{coluc } nz + \Delta \text{ luc } nz.$$

Comparison of these Lucas functions with those derived from the circular functions defined by

$$\sin z = (e^{iz} - e^{-iz})/2i, \qquad \cos z = (e^{iz} + e^{-iz})/2$$

or those from the hyperbolic functions defined similarly by

$$\sinh z = (e^z - e^{-z})/2, \qquad \cosh z = (e^z + e^{-z})/2$$

reveals a close analogy. Also, in the special case that the quadratic equation is $x^2 = x + 1$, we see a familiar list of Fibonacci identities emerging for complex subscripts. This fine reference [2] was brought to our attention by Prof. Tyre A. Newton.

REFERENCES

1. The square root of Q was suggested by Maxey Brooke in a letter.
2. J. C. Amson, "Lucas Functions," Eureka: The Journal of the Archimedeans, (Cambridge University), No. 26, October, 1963, pp. 21-25.
3. S. L. Basin and Verner E. Hoggatt, Jr., "A Primer on the Fibonacci Sequence, Part II," Fibonacci Quarterly, 1:2, pp. 61-68.

xxxxxxxxxxxxxxx

LETTER TO THE EDITOR

P. NAOR
The University of North Carolina
Chapel Hill, N.C.

I read with great interest your recent paper "On the Ordering of the Fibonacci Sequences." The general idea underlying your ordering procedure is excellent, but the representation can be improved and (possibly obscure) relationships may be brought to light.

Consider (for the time being) sequences for which $D \geq 11$. For reasons which will soon become clear I prefer to define the number f_o (in your notation) as the first term in the sequence, ϕ, say. You correctly pointed out that "a negative sequence may be obtained from a positive sequence by changing the signs of all terms"...; however, there is another (rather simple) operation which establishes an equivalence between two sequences. Consider a sequence

$$\cdots\cdots \phi_{-4}, \ \phi_{-3}, \ \phi_{-2}, \ \phi_{-1}, \phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \cdots\cdots$$

and assume, for convenience, that the monotonic portion is positive. It is easy to verify that ϕ_{-n} is positive (negative) if n is even (odd) where n is a non-negative integer. Next view an associated sequence $\{\phi'\}$ defined by

$$\phi'_{\pm n} = \phi_{\mp n} \qquad \text{if } n \text{ is even}$$

$$\phi'_{\pm n} = \phi_{\mp n} \qquad \text{if } n \text{ is odd.}$$

It is elementary to show that $\{\phi'\}$ is a Fibonacci sequence (with the monotonic part positive) - thus Fibonacci sequences typically appear