ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. Any problem believed to be new in the area of recurrent sequences and any new approaches to existing problems will be welcomed. The proposer should submit each problem with solution in legible form, preferably typed in double spacing with name and address of the proposer as a heading.

Solutions to problems listed below should be submitted on separate signed sheets within two months of publication.

B-58 *Proposed by Sidney Kravitz, Dover, New Jersey*

Show that no Fibonacci number other than 1, 2, or 3 is equal to a Lucas number.

B-59 *Proposed by Brother U. Alfred, St. Mary's College, California*

Show that the volume of a truncated right circular cone of slant height F_n with F_{n-1} and F_{n+1} the diameters of the bases is

$$\sqrt{3}\,\pi(F_{n+1}^3 - F_{n-1}^3)/24 .$$

B-60 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that $L_{2n}L_{2n+2} - 5F_{2n+1}^2 = 1$, where F_n and L_n are the n-th Fibonacci number and Lucas number, respectively.

B-61 *Proposed by J.A.H. Hunter, Toronto, Ontario*

Define a sequence $U_1, U_2, \ldots$ by $U_1 = 3$ and

$$U_n = U_{n-1} + n^2 + n + 1 \quad \text{for} \quad n > 1 .$$

Prove that $U_n \equiv 0 \pmod{n}$ if $n \not\equiv 0 \pmod{3}$.

B-62 *Proposed by Brother U. Alfred, St. Mary's College, California*

Prove that a Fibonacci number with odd subscript cannot be represented as the sum of squares of two Fibonacci numbers in more than one way.

B-63 *An old problem whose source is unknown, suggested by Sidney Kravitz, Dover, New Jersey*

In ΔABC let sides AB and AC be equal. Let there be a point D on side AB such that AD = CD = BC. Show that

$$2\cos \measuredangle A = AB/BC = (1 + \sqrt{5})/2 \quad ,$$

the golden mean.

SOLUTIONS

A BOUND ON BOUNDED FIBONACCI NUMBERS

B-44 *Proposed by Douglas Lind, Falls Church, Virginia*

Prove that for every positive integer k there are no more than n Fibonacci numbers between n^k and n^{k+1}.

Solution by the proposer.

Assume the maximum,

$$(1) \qquad n^k < F_{r+1}, F_{r+2}, \ldots, F_{r+n} < n^{k+1} \quad .$$

Now

$$\sum_{j=1}^{n-1} F_{r+j} = \sum_{j=1}^{r+n-1} F_j - \sum_{j=1}^{r} F_j$$

$$= F_{r+n+1} - F_{r+2} \quad .$$

But by (1),

$$\sum_{j=1}^{n-1} F_{r+j} + F_{r+2} > n \cdot n^k$$

and hence

$$F_{r+n+1} > n^{k+1} \quad ,$$

thus proving the proposition.

ANOTHER SUM

B-45 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Let H_n be the n-th generalized Fibonacci number, i.e., let H_1 and H_2 be arbitrary and $H_{n+2} = H_{n+1} + H_n$ for $n > 0$. Show that $nH_1 + (n-1)H_2 + (n-2)H_3 + \ldots + H_n = H_{n+4} - (n+2)H_2 - H_1$.

Solution by David Zeitlin, Minneapolis, Minnesota.

In B-20 (see Fibonacci Quarterly, 2(1964) p. 77), it was shown that

$$\sum_{j=1}^{n} H_j = H_{n+2} - H_2 \quad .$$

In B-40 (see Fibonacci Quarterly, 2(1964), p. 155), Wall proposed that

$$\sum_{j=1}^{n} jH_j = (n+1)H_{n+2} - H_{n+4} + H_1 + H_2 \quad .$$

Thus, the desired sum

$$\sum_{j=1}^{n} \left[(n+1) - j\right] H_j = (n+1) \sum_{j=1}^{n} H_j - \sum_{j=1}^{n} jH_j$$

$$= \left[(n+1)H_{n+2} - (n+1)H_2\right] - \left[(n+1)H_{n+2} - H_{n+4} + H_1 + H_2\right]$$

$$= H_{n+4} - (n+2)H_2 - H_1 \quad .$$

Also solved by Douglas Lind, Kenneth E. Newcomer, Farid K. Shuayto, Sheryl B. Tadlock, Howard L. Walton, Charles Zeigenfus, and the proposer.

A CONTINUANT DETERMINANT

B-46 *Proposed by C.A. Church, Jr., Duke University, Durham, North Carolina*

Evaluate the n-th order determinant

$$D_n = \begin{vmatrix} a+b & ab & 0 & 0 & \cdots \\ 1 & a+b & ab & 0 & \cdots \\ 0 & 1 & a+b & ab & \cdots \\ 0 & 0 & 1 & a+b & \cdots \\ \cdots & & & & \\ \cdots & & & & \\ \cdots & & & & \end{vmatrix} .$$

Solution by F.D. Parker, SUNY, Buffalo, N.Y.

We denote the value of the determinant of order n by D(n), and notice that $D(1) = a + b$ and $D(2) = a^2 + ab + b^2$. Expanding D(n) by the first row, we see that

$$D(n) = (a + b)\, D(n-1) - ab\, D(n-2) \ .$$

This is a homogeneous linear second order difference equation; if $a \neq b$, the solution which fits the initial conditions is

$$D(n) = (a^{n+1} - b^{n+1})/(a-b) \ .$$

If $a = b$, the solution which fits the initial conditions is $D(n) = (1 + n)\, a^n$.

Also solved by Joel L. Brenner, Douglas Lind, C.B.A. Peck, David Zeitlin, and the proposer.

Lind, Peck, and Zeitlin pointed out that B-46 is a special case of B-13. Peck also noted that B-46 is an example of a class of continuants mentioned by J. J. Sylvester in the Philosophical Magazine, Series 4, 5 (1853)446-457. (See T. Muir, History of the Theory of Determinants (Dover) Vol. I, p. 418.) Brenner noted that B-46 and similar problems occur as Nos. 217, 225, 234, etc. in Faddeev and Sominski, Problems in Higher Algebra, a translation of which will soon be published by W. H. Freeman.

CONSECUTIVE COMPOSITE FIBONACCI NUMBERS

B-47 *Proposed by Barry Litvack, University of Michigan, Ann Arbor, Michigan*

Prove that for every positive integer k there are k consecutive Fibonacci numbers each of which is composite.

Solution by Sidney Kravitz, Dover, New Jersey

Let F_n be the n-th Fibonacci number. We note that $F_n > 1$ for $n > 2$, that F_j divides F_{mj} and that j is a divisor of $(k+2)! + j$ for $3 \leq j \leq k+2$. Thus the k consecutive Fibonacci numbers

$$F_{(k+2)!+3},\ F_{(k+2)!+4},\ \ldots,\ F_{(k+2)!+k+2}$$

are divisible by $F_3, F_4, \ldots, F_{k+2}$ respectively.

Also solved by R.W. Castown, Douglas Lind, F.D. Parker, and the proposer.

A BINOMIAL EXPANSION

B-48 *Proposed by H.H. Ferns, University of Victoria, Victoria, British Columbia, Canada*

Prove that

$$\sum_{k=1}^{r-1} (-2)^k \binom{r}{k} F_k = \begin{cases} -2^r F_r & \text{if } r \text{ is an even positive integer} \\ 2^r F_r - 2(5)^{(r-1)/2} & \text{if } r \text{ is an odd positive integer,} \end{cases}$$

where $F_{n+2} = F_{n+1} + F_n (F_1 = F_2 = 1)$ and find the corresponding sum in which the F_k are replaced by the Lucas numbers L_k.

Solution by D.G. Mead, University of Santa Clara, Santa Clara, California

Let S be the given sum. By the Binet formula,

$$F_n = (a^n - b^n)/(a-b)$$

where $a = (1 + \sqrt{5})/2$ and $b = 1 - a$. Then $a - b = \sqrt{5} = 1 - 2a = 2b - 1$, and

$$S + (-2)^r F_r = \sum_{k=0}^{r} \binom{r}{k} (-2)^k F_k$$

$$= \frac{1}{\sqrt{5}} \sum_{k=0}^{r} \binom{r}{k} \left[(-2a)^k - (-2b)^k\right]$$

$$= \frac{(1 - 2a)^r - (1 - 2b)^r}{\sqrt{5}}$$

$$= \frac{(\sqrt{5})^r \left[1 - (-1)^r\right]}{\sqrt{5}} .$$

The desired conclusion follows immediately.

Similarly one sees from $L_n = a^n + b^n$ that the corresponding sum for the Lucas numbers is $-2 - 2^r L_r + 2(\sqrt{5})^r$ for r even and $-2 + 2^r L_r$ for r odd.

Also solved by the proposer.

AN ALPHAMETIC

B-49 *Proposed by Anton Glaser, Pennsylvania State University, Abington, Pennsylvania*

Let ϕ represent the letter "oh". Given that T, W, ϕ, L, V, P, and TWϕ are Fibonacci numbers, solve the cryptarithm in the base 14, introducing the digits α, β, γ, and δ in base 14 for 10, 11, 12, and 13 in base 10.

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  TWφ
   IS
  THE
 φNLY
 EVEN
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PRIME
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Solution by Charles Ziegenfus, Madison College, Harrisonburg, Virginia

With a little calculation one observes that the Fibonacci number corresponding to TWϕ is 2584. Thus, $T = \delta$, $W = 2$, $\phi = 8$, $P = 1$, $L = 3$ or 5, and $V = 3$ or 5. Next we note that $8 + E + (2 \text{ or } 3) = 1R$, so that $E = 4 + R$ or $E = 3 + R$. Tabulating these results:

R	0	6	7	4	6	7	9
E	4	α	β	7	9	α	γ

Further, $8 + S + E + Y + N = kE$ or $S + Y + N = k0 - 8 = 6$, 16, or 26 in base 14. There are no possible choices for S, Y, N such that $S + Y + N = 6$ or 26. Thus S, Y, N can be chosen from $\{0, 9, \beta\}$; $\{4, 7, 9\}$; $\{4, 6, \alpha\}$. Tabulating this with the previous result we obtain:

R	6	7	4	7	9
E	α	β	7	α	γ
S, Y, N	4, 7, 9 or 0, 9, β	4, 6, α	0, 9, β	0, 9, β	4, 6, α

Further,

$$T + W + \phi + I + S + T + H + E + \phi + N + L + Y + E + V + E + N$$

$$- P - R - I - M - E \text{ is a multiple of } \delta .$$

We reduce the above to $6 + (2 \cdot E - R) + H + N - M = \delta \cdot k$.

On substituting the possible values for R and E we further reduce this problem to the following cases:

a. $R = 6$ and $E = \alpha$, $7 + N + H - M = \delta \cdot k$.
b. $R = 7$ and $E = \beta$, $8 + N + H - M = \delta \cdot k$.
c. $R = 4$ and $E = 7$, $3 + N + H - M = \delta \cdot k$.
d. $R = 7$ and $E = \alpha$, $6 + N + H - M = \delta \cdot k$.
e. $R = 9$ and $E = \gamma$, $8 + H + N - M = \delta \cdot k$.

From the previous table we observe that there are exactly three choices for N. Using these in the above cases reduces the problem to an equation involving only H and M and only three choices for these. Thus we obtain two distinct solutions (actually four since S and Y can be interchanged).

0	1	2	3	4	5	6	7	8	9	α	β	γ	δ
S or Y	P	W	V	H	L	R	M	ϕ	N	E	S or Y	I	T
M	P	W	V	S or Y	L	R	N	ϕ	S or Y	E	I	H	T

Also solved by the proposer and partially solved by J.A.H. Hunter.

AND ANOTHER SUM

B-50 *Proposed by Douglas Lind, Falls Church, Virginia*

Prove that

$$\sum_{j=0}^{n} \left[2F_j^2 - \binom{n}{j} F_j\right] = F_n^2 .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Since

$$\sum_{j=0}^{n} F_j^2 = F_n F_{n+1},$$

$$\sum_{j=0}^{n} \binom{n}{j} F_j = F_{2n} = F_n L_n = F_n(F_{n+1} + F_{n-1}),$$

the desired sum is

$$2F_n F_{n+1} - F_n F_{n+1} - F_n F_{n-1} = F_n(F_{n+1} - F_{n-1}) = F_n^2 .$$

Also solved by H.H. Ferns, Farid K. Shuayto, Sheryl B. Tadlock, and the proposer.

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