# ON A GENERAL FIBONACCI IDENTITY

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1. The Fibonacci sequence is defined by the recurrence relation

(1) $$F_{n+2} = F_{n+1} + F_n \quad ,$$

together with the particular values

(2) $$F_0 = 0, \ F_1 = 1 \quad .$$

It is easily verified that the unique solution* of (1) and (2) is given by

(3) $$F_n = (\alpha^n - \beta^n)/(\alpha - \beta) \quad ,$$

where $\alpha$ and $\beta$ are the roots of the equation

(4) $$x^2 = x + 1 \quad ,$$

namely

(5) $$\alpha = \frac{1}{2}(1 + \sqrt{5}), \ \beta = \frac{1}{2}(1 - \sqrt{5}) = -\alpha^{-1} \quad .$$

The sequence is thus defined for all integers n, positive or negative or zero. From (1) and (2), we infer that (3) takes integer values for all n, and we observe, by (3) and (5), that

(6) $$F_{-n} = (-1)^{n+1} F_n \quad .$$

This sequence and its generalizations have been the subject of a vast literature, and a very large number of identities of different kinds, involving the Fibonacci numbers, can be demonstrated. It is the purpose of this paper to show how a considerable body of these may be obtained as particular cases of a single identity.

*Direct substitution shows that (3) is a solution of (1) and (2). If $F'_n$ were another solution, $f_n = F_n - F'_n$ would satisfy a relation (1), with $f_0 = f_1 = 0$. Induction on n now shows that $f_n = 0$ for all n, so that (3) is the unique solution, as stated.

2. We begin by defining the function

(7) $$S_0(n) = F_n + F_{n+1} - F_{n+2} \quad .$$

Then, immediately, by (1), we see that, for all integers n,

(8) $$S_0(n) = 0 \quad .$$

Now consider the function

(9) $$S_1(m, n) = F_m F_n + F_{m+1} F_{n+1} - F_{m+n+1} \quad .$$

Then, by (1), for any m and n,

(10) $$S_1(m + 1, n) = S_1(m, n) + S_1(m - 1, n) \quad .$$

Also, by (1), (2), (7), and (8), we have that

(11) $$\begin{cases} S_1(0, n) = F_{n+1} - F_{n+1} = 0 \\ \\ S_1(1, n) = S_0(n) = 0; \end{cases}$$ and

whence (10) yields, by upward and downward induction on m, that, for all integers m and n,*

(12) $$S_1(m, n) = 0 \quad .$$

Next consider the function

(13) $$S_2(r, m, n) = F_m F_n - (-1)^r (F_{m+r} F_{n+r} - F_r F_{m+n+r}) \quad .$$

Again applying (1), we see that

(14) $$S_2(r + 1, m, n) = S_2(r - 1, m, n+2) - S_2(r, m, n + 1) \quad .$$

Now, for any m or n, by (2), (9), and (12),

(15) $$\begin{cases} S_2(0, m, n) = F_m F_n - F_m F_n = 0 \\ \\ S_2(1, m, n) = S_1(m, n) = 0 \quad . \end{cases}$$ and

*We may also note that, for any fixed n, (10) is a relation of the form (1). Thus, as in the previous footnote, we get (12), for all m and n.

Thus, by upward and downward induction on $r$ in (14),* we find that, for all integers $r$, m, and n,

$$S_2(r, m, n) = 0 \quad . \tag{16}$$

Finally consider the function (with $k \geq 0$)

$$S_3(k, r, m, n) = F_m^k F_n - (-1)^{kr} \sum_{h=0}^{k} \binom{k}{n} (-1)^h F_r^h F_{r+m}^{k-h} F_{n+kr+hm} \quad . \tag{17}$$

It is well known that

$$\binom{k+1}{h} = \binom{k}{h} + \binom{k}{h-1} \tag{18}$$

and

$$\binom{k}{h} = 0 \text{ when } h < 0 \text{ or } 0 \leq k < h \quad . \tag{19}$$

Thus we can show, by (13), (16), (18), and (19), that

$$S_3(k+1, r, m, n) = F_m S_3(k, r, m, n) \quad . \tag{20}$$

Also, by (13), (16), (17), and (19),

$$\left\{ \begin{aligned} & S_3(0, r, m, n) = F_n - F_n = 0 \\ \text{and} \quad & S_3(1, r, m, n) = S_2(r, m, n) = 0 \quad . \end{aligned} \right. \tag{21}$$

Thus, by upward induction on k in (20), we get that, for all integers $r$, m, and n, and all integers $k \geq 0$,

$$S_3(k, r, m, n) = 0 \quad . \tag{22}$$

*We observe that, while the inductive argument leading to (12) assumes an arbitrarily chosen and fixed n; the corresponding argument yielding (16) assumes, at each step, that (16) holds for a consecutive pair of values of $r$, an arbitrary fixed value of m, and <u>all</u> values of n.

This is the general identity promised above:

$$(23) \qquad \boxed{F_m^k F_n = (-1)^{k\tau} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_\tau^h F_{\tau+m}^{k-h} F_{n+k\tau+hm}}.$$

3. We may now look at some of the identities which are obtained as particular cases of (23). On the left of each identity below, the vector $(k, \tau, m, n)$ is shown. In some cases, the identity (6) is used to remove negative subscripts.

$$(24) \quad (k, \tau, -m, -n): F_m^k F_n = \sum_{h=0}^{k} \binom{k}{h} (-1)^{(m-1)(k-h)} F_\tau^h F_{\tau-m}^{k-h} F_{n-k\tau+hm}$$

$$(25) \quad (k, \tau, -m, -k\tau-n): F_m^k F_{n+k\tau} = \sum_{h=0}^{k} \binom{k}{h} (-1)^{(m-1)(k-h)} F_\tau^h F_{\tau-m}^{k-h} F_{n+hm}$$

$$(26) \quad (k, \tau, m, -k\tau): F_m^k F_{k\tau} = \sum_{h=1}^{k} \binom{k}{h} (-1)^{h-1} F_\tau^h F_{\tau+m}^{k-h} F_{hm}$$

$$(27) \quad (k, \tau, m, m): F_m^{k+1} = (-1)^{k\tau} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_\tau^h F_{\tau+m}^{k-h} F_{k\tau+(h+1)m}$$

$$(28) \quad (k, \tau, m, n\tau): F_m^k F_{n\tau} = (-1)^{k\tau} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_\tau^h F_{\tau+m}^{k-h} F_{(n+k)\tau+hm}$$

$$(29) \quad (k, \tau, m\tau, n): F_{m\tau}^k F_n = (-1)^{k\tau} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_\tau^h F_{(m+1)\tau}^{k-h} F_{n+(k+hm)\tau}$$

$$(30) \quad (k, \tau, m, 0): \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_\tau^h F_{\tau+m}^{k-h} F_{k\tau+hm} = 0$$

$$(31) \quad (k, \tau, m, \pm 1): F_m^k = (-1)^{k\tau} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_\tau^h F_{\tau+m}^{k-h} F_{k\tau+hm\pm 1}$$

(32) $(k, r, mr, 0)$: $$\sum_{h=0}^{k} \binom{k}{h}(-1)^h F_r^h F_{(m+1)r}^{k-h} F_{(k+hm)r} = 0$$

(33) $(k, r, \pm 1, n)$: $$F_n = (-1)^{kr} \sum_{h=0}^{k} \binom{k}{h}(-1)^h F_r^h F_{r\pm 1}^{k-h} F_{n+kr\pm h}$$

(34) $(k, r, \pm 1, -kr-n)$: $$F_{n+kr} = \sum_{h=0}^{k} \binom{k}{h} F_r^h F_{r\pm 1}^{k-h} F_{n\mp h}$$

(35) $(k, r, \pm 1, -kr)$: $$F_{kr} = \sum_{h=1}^{k} \binom{k}{h} F_r^h F_{r\pm 1}^{k-h} F_{\mp h}$$

(36) $(k, r, \pm 1, nr)$: $$F_{nr} = (-1)^{kr} \sum_{h=0}^{k} \binom{k}{h}(-1)^h F_r^h F_{r\pm 1}^{k-h} F_{(n+k)r\pm h}$$

(37) $(k, r, \pm 2, n)$: $$F_n = (\pm 1)^k (-1)^{kr} \sum_{h=0}^{k} \binom{k}{h}(-1)^h F_r^h F_{r\pm 2}^{k-h} F_{n+kr\pm 2h}$$

(38) $(k, r, \pm 2, -kr)$: $$F_{kr} = (\pm 1)^k \sum_{h=1}^{k} \binom{k}{h}(-1)^{h-1} F_r^h F_{r\pm 2}^{k-h} F_{\pm 2h}$$

(39) $(k, \pm 1, m, n)$: $$F_m^k F_n = (-1)^k \sum_{h=0}^{k} \binom{k}{h}(-1)^h F_{m\pm 1}^{k-h} F_{n\pm k+hm}$$

(40) $(k, \pm 1, m, \mp k)$: $$F_m^k F_k = (\pm 1)^k \sum_{h=1}^{k} \binom{k}{h}(-1)^{h-1} F_{m\pm 1}^{k-h} F_{hm}$$

(41) $(k, \pm 2, m, n)$: $$F_m^k F_n = \sum_{h=0}^{k} \binom{k}{h}(\mp 1)^h F_{m\pm 2}^{k-h} F_{n\pm 2k+hm}$$

(42) $(k, r, \pm 1, 0)$: $$\sum_{h=0}^{k} \binom{k}{h}(-1)^h F_r^h F_{r\pm 1}^{k-h} F_{kr\pm h} = 0$$

(43) $(k, r, \pm 2, 0)$: $$\sum_{h=0}^{k} \binom{k}{h}(-1)^h F_r^h F_{r\pm 2}^{k-h} F_{kr\pm 2h} = 0$$

(44) $(k, \pm 1, m, 0)$: $$\sum_{h=0}^{k} \binom{k}{h}(-1)^h F_{m\pm 1}^{k-h} F_{hm\pm k} = 0$$

(45) $(k, \pm 1, m, \pm' 1)$: $$F_m^k = (-1)^k \sum_{h=0}^{k} \binom{k}{h}(-1)^h F_{m\pm 1}^{k-h} F_{hm\pm k\pm' 1}$$

(46) $(k, 1, 1, -n)$: $$F_n = \sum_{h=0}^{k} \binom{k}{h} F_{n-k-h}$$

(47) $(k, 1, 1, -nk)$: $$F_{nk} = \sum_{h=0}^{k} \binom{k}{h} F_{(n-1)k-h}$$

(48) $(k, 1, 1, -k)$: $$F_k = \sum_{h=0}^{k} \binom{k}{h}(-1)^{h-1} F_h$$

(49) $(k, 2, -1, -n)$: $$F_n = \sum_{h=0}^{k} \binom{k}{h} F_{n-2k+h}$$

(50) $(k, -1, 2, -n)$: $$F_n = \sum_{h=0}^{k} \binom{k}{h}(-1)^h F_{n+k-2h}$$

(51) $(k, 2, -1, -2k)$: $$F_{2k} = \sum_{h=0}^{k} \binom{k}{h} F_h$$

(52) (k, -1, 2, k): $F_k = \sum_{h=0}^{k} \binom{k}{h}(-1)^{k-h} F_{2h}$

(53) (k, 1, 1, 0): $\sum_{h=0}^{k} \binom{k}{h}(-1)^{h} F_{k+h} = 0$

(54) (k, 2, -1, 0): $\sum_{h=0}^{k} \binom{k}{h}(-1)^{h} F_{2k-h} = 0$

(55) (k, -1, 2, 0): $\sum_{h=0}^{k} \binom{k}{h}(-1)^{h} F_{k-2h} = 0$

(56) $(1, r, m, n)$: $F_m F_n = (-1)^r (F_{m+r} F_{n+r} - F_r F_{m+n+r})$

(57) $(1, r, m, -n)$: $F_m F_n - F_{m+r} F_{n-r} = (-1)^{n-r} F_r F_{m-n+r}$

(58) $(1, r, m, -m)$: $F_m^2 - F_{m+r} F_{m-r} = (-1)^{m-r} F_r^2$

(59) $(1, r, m, n-r)$: $F_r F_{m+n} = F_{r+m} F_n - (-1)^r F_m F_{n-r}$

(60) $(1, r-k, m+k, -m+k)$: $F_{m+k} F_{m-k} - F_{m+r} F_{m-r} = (-1)^{m-r} F_r F_{k+r}$

(61) $(1, \pm 1, m, n)$: $F_{m+n\pm 1} = F_m F_n + F_{m\pm 1} F_{n\pm 1}$

(62) (1, 1, m, n-1): $F_{m+n} = F_{m+1} F_n + F_m F_{n-1}$

(63) (1, 2, m-1, n-1): $F_{m+n} = F_{m+1} F_{n+1} - F_{m-1} F_{n-1}$

(64) $(1, \pm 1, m, m)$: $F_{2m\pm 1} = F_m^2 + F_{m\pm 1}^2$

(65) (1, 1, m, m-1): $F_{2m} = F_m (F_{m+1} + F_{m-1})$

(66) (1, 2, m-1, m-1): $F_{2m} = F_{m+1}^2 - F_{m-1}^2$

(67) (1, 1, m, -m): $F_m^2 - F_{m+1}F_{m-1} = (-1)^{m-1}$

(68) (1, 2, m, -m): $F_m^2 - F_{m+2}F_{m-2} = (-1)^m$

(69) (1, 1, m+1, -m+1): $F_{m+1}F_{m-1} - F_{m+2}F_{m-2} = 2(-1)^m$

This rather long list includes most of the identities, derivable as special cases of (23), which I have found in the literature, and a number of others (including (23) itself, (24)-(32), (37)-(45), and (60)), which I believe to be new and useful.*

4. We may now ask what else can be done with the family of identities (23)-(69). Some of the further developments will be demonstrated below.

Putting n = m in (59) and dividing by $F_m$, we obtain, by (65) and (6), that

(70) $$(F_{m+1} + F_{m-1})F_r = F_{r+m} + (-1)^m F_{r-m} \quad .$$

Thus

(71) $$\left[F_{m+1} + F_{m-1} - 1 - (-1)^m\right] F_r = (F_{r+m} - F_r) - (-1)^m(F_r - F_{r-m}) \quad .$$

The usefulness of this identity is seen when we put $r = rm + n$ and sum from r = 1 to r = t. The right-hand side telescopes to yield

(72) $$\sum_{r=1}^{t} F_{rm+n} = \frac{F_{(t+1)m+n} - F_{m+n} - (-1)^m(F_{tm+n} - F_n)}{F_{m+1} + F_{m-1} - 1 - (-1)^m}$$

(This result is known [1], but I believe that the line of proof is new.) Certain particular cases have been known for a long time; for instance,

*EDITORIAL NOTE: A different form of the identity (23) appears in an unpublished Master's Thesis entitled "Moduls m properties of the Fibonacci numbers," written by John Vinson at Oregon State University in 1961. (Other parts of that thesis appear as a paper by John Vinson in the Fibonacci Quarterly, 1(1963) No. 2, pp. 37-45.)

$$(73)\quad \sum_{r=1}^{t} F_{r+s} = F_{t+1+2} - F_{1+s} + F_{t+s} - F_s = F_{t+s+2} - F_{s+2} ,$$

$$(74)\quad \sum_{r=1}^{t} F_{2(r+s)} = F_{2(t+1+s)} - F_{2(1+s)} - F_{2(t+s)} + F_{2s} = F_{2(t+s)+1} - F_{2s+1} ,$$

$$(75)\quad \sum_{r=1}^{t} F_{2(r+s)-1} = F_{2(t+s)} - F_{2s} ,$$

and

$$(76)\quad \sum_{r=1}^{t} F_{3(r+s)} = \frac{1}{4}(F_{3(t+1+s)} - F_{3(1+s)} + F_{3(t+s)} - F_{3s}) = \frac{1}{2}(F_{3(t+s)+2} - F_{3s+2}) .$$

If we sum (64) from $m = s + 1$ to $m = s + t$, put $r = m - s$, use (75), and slightly rearrange the result, we obtain that

$$(77)\quad \sum_{r=1}^{t} F_{r+s}^2 = \frac{1}{2}(F_{2(t+s)} + F_{t+s}^2 - F_{2s} - F_s^2) .$$

Now rewrite (65), using (1), in the form

$$(78)\quad F_{2m} \pm F_m^2 = 2F_m F_{m\pm 1} ,$$

and sum (78) as before, using (74) and (77); then we get

$$(79)\quad \sum_{r=1}^{t} F_{r+s} F_{r+s+1} = \frac{1}{4}(F_{2(t+s)+3} + F_{t+s}^2 - F_{2s+3} - F_s^2) .$$

If we sum (73) with $t = w - s$, from $s = v$ to $s = w - 1$, we get that

$$\sum_{s=v}^{w-1} \sum_{u=s+1}^{w} F_u = \sum_{u=v+1}^{w} \sum_{s=v}^{u-1} F_u = \sum_{u=v+1}^{w} (u-v)F_u = \sum_{u=v+1}^{w} uF_u + vF_{v+2} -$$

$$- vF_{w+2} = \sum_{s=v}^{w-1} (F_{w+2} - F_{s+2}) = (w - v)F_{w+2} - F_{w+3} + F_{v+3} ,$$

which yields

$$(80) \qquad \sum_{u=v+1}^{w} uF_u = wF_{w+2} - F_{w+3} - vF_{v+2} + F_{v+3} .$$

The same process of summation applied to (80) yields

$$(81) \quad \sum_{u=v+1}^{w} u^2F_u = w^2F_{w+2} - (2w-1)F_{w+3} + 2F_{w+4} + (2v-1)F_{v+3} - 2F_{v+4}$$

and we can evidently iterate the procedure to obtain the sum of $u^m F_u$ for any positive integer $m$.

Again, replace $m$ by $r + m$ in (63) and apply (61) to the result. This gives

$$F_{r+m+n} = (F_{r+1}F_{m+1} + F_r F_m)F_{n+1} - (F_r F_m + F_{r-1}F_{m-1})F_{n-1}$$

or, by (1),

$$(82) \quad F_{r+m+n} = F_{r+1}F_{m+1}F_{n+1} + F_r F_m F_n - F_{r-1}F_{m-1}F_{n-1} .$$

In particular,

$$(83) \qquad F_{3m} = F_{m+1}^3 + F_m^3 - F_{m-1}^3$$

and

$$(84) \quad F_{3m} = F_m F_{m+1} F_{m+2} + F_{m-1}F_m F_{m+1} - F_{m-2}F_{m-1}F_m .$$

We may note, at this point, that (83) can be put in yet another form, with the help of (67):

$$F_{3m} = F_m^3 + (F_{m+1} - F_{m-1})(F_{m+1}^2 + F_{m+1}F_{m-1} + F_{m-1}^2) =$$

$$= F_m^3 + F_m\left[(F_{m+1} - F_{m-1})^2 + 3F_{m+1}F_{m-1}\right] =$$

$$= F_m^3 + F_m\left\{F_m^2 + 3\left[F_m^2 + (-1)^m\right]\right\}$$

or

(85) $$F_{3m} = 5F_m^3 + 3(-1)^m F_m \quad .$$

By summing (83) and (84) from $m = s + 1$ to $m = s + t$, and using (76), we obtain respectively that

(86) $$\sum_{r=1}^{t} F_{r+s}^3 = \frac{1}{2}(F_{3(t+s)-1} - 2F_{t+s-1}^3 - F_{3s-1} + 2F_{s-1}^3)$$

and

(87) $$\sum_{r=1}^{t} F_{r+s-1}F_{r+s}F_{r+s+1} = \sum_{r=1}^{t} F_{r+s}^3 + (-1)^{t+s}F_{t+s-1} - (-1)^s F_{s-1} \quad .$$

If we multiply (67) by $F_m$ and sum from $m = s + 1$ to $m = s + t$, we get that

$$\sum_{r=1}^{t} F_{r+s}^3 - \sum_{r=1}^{t} F_{r+s-1}F_{r+s}F_{r+s+1} = \sum_{r=1}^{t} (-1)^{r+s-1}F_{r+s} \quad .$$

A comparison of this last result with (87) yields

(88) $$\sum_{r=1}^{t} (-1)^{r+s}F_{r+s} = (-1)^{t+s}F_{t+s-1} - (-1)^s F_{s-1} \quad .$$

This last result may be verified by combining (73) and (74), or by summing the identity (derived from (1)),

(89) $$(-1)^{r+s}F_{r+s} = (-1)^{r+s-2}F_{r+s-2} - (-1)^{r+s-1}F_{r+s-1} \quad .$$

As a final illustration of the large family of identities springing from (23), we consider the generalizations of (66) and (83), analogous to that of (1) in (70). First we obtain a few results analogous to (85). Clearly

$$(F_{m+1} + F_{m-1})^2 = (F_{m+1} - F_{m-1})^2 + 4F_{m+1}F_{m-1} \quad ;$$

thus, by (1) and (67),

$$(F_{m+1} + F_{m-1})^2 = 5F_m^2 + 4(-1)^m \quad , \tag{90}$$

and therefore, by (65),

$$F_{2m}^2 = 5F_m^4 + 4(-1)^m F_m^2 \quad . \tag{91}$$

Also, by (1),

$$F_{n+1} + F_{n-1} = F_{n+2} - F_{n-2} = \frac{1}{2}(F_{n+3} + F_{n-3}) \quad ; \tag{92}$$

whence, by (85) and (67),

$$F_{3m+1} + F_{3m-1} = (F_{m+1} + F_{m-1})\left[5F_m^2 + (-1)^m\right] \quad . \tag{93}$$

Putting $2r$ for $r$ and 2m for m in (70), we get, by (64), (66), and (67), that

$$\left[5F_m^2 + 2(-1)^m\right]F_{2r} = F_{r+m+1}^2 - F_{r+m-1}^2 + F_{r-m+1}^2 - F_{r-m-1}^2 \quad . \tag{94}$$

Alternatively, on squaring (70), we obtain, by (58) and (90), that

$$\left[5F_m^2 + 4(-1)^m\right]F_r^2 = F_{r+m}^2 + F_{r-m}^2 + 2(-1)^m\left[F_r^2 - (-1)^{r+m}F_m^2\right] \quad ,$$

whence

$$\left[5F_m^2 + 2(-1)^m\right]F_r^2 + 2(-1)^r F_m^2 = F_{r+m}^2 + F_{r-m}^2 \quad . \tag{95}$$

We see that (94) yields (66) on putting one for m and m for $r$. Finally, put $3r$ for $r$ and 3m for m in (70). Then, by (85) and (93), we get

$$(F_{m+1} + F_{m-1})\left[5F_m^2 + (-1)^m\right]F_{3r} = 5F_{r+m}^3 + 3(-1)^{r+m}F_{r+m} +$$
$$+ 5(-1)^m F_{r-m}^3 + 3(-1)^r F_{r-m}$$
$$= 5F_{r+m}^3 + 5(-1)^m F_{r-m}^3 + 3(-1)^{r+m}(F_{m+1} + F_{m-1})F_r$$
$$= 5F_{r+m}^3 + 5(-1)^m F_{r-m}^3 + (-1)^m (F_{m+1} + F_{m-1})(F_{3r} - 5F_r^3) \quad .$$

Thus, by (65),

(96) $$F_m F_{2m} F_{3r} = F_{r+m}^3 - (-1)^m (F_{m+1} + F_{m-1})F_r^3 + (-1)^m F_{r-m}^3 \quad .$$

This identity is new, but we can find in the literature [2] the particular cases when $m = 1$ and $m = 2$, namely (83) (with $r$ for $m$) and

(97) $$3F_{3r} = F_{r+2}^3 - 3F_r^3 + F_{r-2}^3 \quad .$$

## REFERENCES

1. K. Siler, "Fibonacci Summations," Fibonacci Quarterly, 1(1963) No. 3, pp. 67-69.

2. Fibonacci Quarterly, 1(1963) No. 2, p. 60.

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