SEEKING THE LOST GOLD MINE OR EXPLORING FOR FIBONACCI FACTORIZATIONS

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Now that summer is coming on, everybody is looking for a good way to waste time. Seek no farther. The search for factors of Fibonacci numbers is the perfect answer.

And first, some ground rules. People with computers who program their machines and then sit idly by while they grind out answers should not be considered in the class of working Fibonacci factorizers. The challenge is to be able with the available tables and the mathematical bow-and-arrow — the calculator — to find some method or methods that facilitate the determination of factors in Fibonacci sequences.

Just to get away from the well-worn path we start in virgin territory with a sequence 1, 4, 5, 9, 14, 23, etc. We discover very soon that this has a prolongation to the left of ....-19, 12, -7, 5, -2, 3, 1, 4, 5, 9....... and since the factors of both portions of the sequence are the same we might call the sequence 2, 5, 7, 12, etc., the conjugate sequence to 1, 4, 5, 9, 14, etc. This is a first help in factoring the initial hundred terms of each portion of our sequence — a not too modest goal.

Next, we can determine the primes that do not divide the members of our sequence. Taking the square of any term minus the product of the two adjacent terms gives ± 11. Thus

$$5^2 - 4 \cdot 9 = -11$$

In general, if we designate the terms of the sequence T_n,

$$T_n^2 - T_{n-1} T_{n+1} = \pm 11$$

Hence if a prime divides T_{n-1}, for example, it would follow that

$$T_n^2 \equiv \pm 11 \pmod{p}$$

Thus if neither +11 nor -11 is a quadratic residue of a given prime p, then this prime cannot be a factor of the sequence. We can eliminate from consideration in this way: 11, 13, 17, 29, 41, 61, etc.

The smaller quantities in our sequence can be factored either by inspection or factor tables. Next, apart from 11, the primes have the same period in as in the Fibonacci sequence. Hence we can have some idea of when they should be entering the sequence by looking at the size of the period and more specifically the entry point in the Fibonacci sequence. For the spacing of the members of our sequence that are divisible by the given prime is the same as in the Fibonacci sequence should it be a factor of the sequence at all. For small spacings we can then extend the factor to other members of our sequence by using this information regarding the period and entry point of the prime in the Fibonacci sequence.

But how should we organize a systematic and convenient method of factoring using previous information on the Fibonacci and Lucas sequences? The following approach was tried. Since

$$T_1 = 1 = F_0 + L_1$$

$$T_2 = 4 = F_1 + L_2$$

it follows in general that $T_n = F_{n-1} + L_n$. Thus if we know the Fibonacci numbers modulo p and the Lucas numbers modulo p, it is simply necessary to check and see whether the sum of the residues of F_{n-1} and L_n is congruent to zero modulo p. Another dividend comes from the fact that if we call the members of the conjugate sequence R_n then

$$R_1 = 2 = F_2 + L_1$$

$$R_2 = 5 = F_3 + L_2$$

so that in general $R_n = F_{n+1} + L_n$. Thus the residues can be used in two ways. The original thought was that once these residues are on hand, it would be possible to use them for factoring many Fibonacci sequences.

The method works. But — as we get to larger and larger primes the periods increase and so likewise do the entry points so that the probability that the prime will be a factor between T_{100} and R_{100} gets less. Also, with large primes such as 911 with an entry point of 70 in the Fibonacci sequence (1, 1, 2, 3...) the probability that this will be a factor of a Fibonacci sequence chosen at random is relatively small, being only 7.6%. This same pattern applies to all large primes with relatively small entry points.

Again the sequence of primes that factor all Fibonacci sequences have the maximum period, $2p + 2$ and hence tend to have a small probability of factoring our sequences within the limited range from T_{100} to R_{100}.

All in all, the high hopes entertained for this method were not realized. Does some one have a better way of attacking this problem?

As a byproduct, it would appear to be a worthwhile goal to have available factorizations of the first hundred terms of a few Fibonacci sequences such as (1, 4) and (2, 5) — even if somebody does it on a computer.

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