ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-61 *Proposed by P. F. Byrd, San Jose State College, San Jose, California (corrected)*

Let

$$f_{n,k} = 0 \quad \text{for} \quad 0 \le n \le k-2, \quad f_{k-1,k} = 1 \quad \text{and}$$

$$f_{n,k} = \sum_{j=1}^{k} f_{n-j,k} \quad \text{for} \quad n \ge k \; .$$

Show that

$$\frac{1}{2} < \frac{f_{n,k}}{f_{n+1,k}} < \frac{1}{2} + \frac{1}{2k} \quad \text{as} \quad n \longrightarrow \infty \; .$$

Hence

$$\lim_{k \longrightarrow \infty} \; \lim_{n \longrightarrow \infty} \; \frac{f_{n,k}}{f_{n+1,k}} = \frac{1}{2} \; .$$

H-65 *Proposed by J. Wlodarski, Porz-Westhoven, Federal Republic of Germany*

The units digit of a positive integer, M, is 9. Take the 9 and put it on the left of the remaining digits of M forming a new integer, N, such that N = 9M. Find the smallest M for which this is possible.

H-66 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va., and Raymond Whitney, Pennsylvania State University, Hazelton Campus, Hazelton, Pa.*

Let

$$\sum_{j=0}^{k} a_j y_{n+j} = 0$$

be a linear homogeneous recurrence relation with constant coefficients a_j. Let the roots of the auxiliary polynomial

$$\sum_{j=0}^{k} a_j x^j = 0 \quad \text{be} \quad r_1, r_2, \ldots, r_m$$

and each root r_i be of multiplicity m_i $(i = 1, 2, \ldots, m)$. Jeske (Linear Recurrence Relations - Part 1, Fibonacci Quarterly, Vol. 1, No. 2, pp. 69-74) showed that

$$\sum_{n=0}^{\infty} y_n \frac{t^n}{n!} = \sum_{i=1}^{m} e^{r_i t} \sum_{j=0}^{m_i - 1} b_{ij} t^j .$$

He also stated that from this we may obtain

$$(*) \qquad y_n = \sum_{i=1}^{m} r_i^n \sum_{j=0}^{m_i - 1} b_{ij} n^j .$$

(i) Show that (*) is in general incorrect, (ii) state under what conditions it yields the correct result, and (iii) give the correct formulation.

H-67 *Proposed by J. W. Gootherts, Sunnyvale, California*

Let $B = (B_0, B_1, \ldots B_n)$ and $V = (F_m, F_{m+1}, \ldots F_{m+n})$ be two vectors in Euclidian $n + 1$ space. The B_i's are binomial coefficients of degree n and the F_{m+i}'s are consecutive Fibonacci numbers starting at any integer m.

Find the limit of the angle between these vectors as n approaches infinity.

H-68 *Proposed by H. W. Gould, West Virginia University, Morgantown, West Virginia*

Prove that

$$\sum_{k=1}^{n} \frac{1}{F_k} \geq \frac{n^2}{F_{n+2}-1}, \quad n \geq 1$$

with equality only for $n = 1, 2$.

H-62 *Proposed by H. W. Gould, West Virginia University, Morgantown, West Virginia (corrected)*

Find all polynomials $f(x)$ and $g(x)$, of the form

$$f(x+1) = \sum_{j=0}^{r} a_j x^j, \quad a_j \text{ an integer}$$

$$g(x) = \sum_{j=0}^{s} b_j x^j, \quad b_j \text{ an integer}$$

such that

$$2\left\{x^2 f^3(x+1) - (x+1)^2 g^3(x)\right\} + 3\left\{x^2 f^2(x+1) - (x+1)^2 g^2(x)\right\}$$
$$+(2x+1)\left\{xf(x+1) - (x+1)g(x)\right\} = 0 .$$

H-69 *Proposed by M. N. S. Swamy, University of Saskatchewan, Regina, Canada*

Given the polynomials $B_n(x)$ and $b_n(x)$ defined by,

$$b_n(x) = x\,B_{n-1}(x) + b_{n-1}(x) \quad (n > 0)$$
$$B_n(x) = (x+1)\,B_{n-1}(x) + b_{n-1}(x) \quad (n > 0)$$
$$b_0(x) = B_0(x) = 1$$

it is possible to show that,

$$B_n(x) = \sum_{r=0}^{n} \binom{n+r+1}{n-r} x^r ,$$

and

$$b_n(x) = \sum_{r=0}^{n} \binom{n+r}{n-r} x^r .$$

It can also be shown that the zeros of $B_n(x)$ or $b_n(x)$ are all real, negative and distinct. The problem is whether it is possible to factorize $B_n(x)$ and $b_n(x)$. I have found that for the first few values of n, the result

$$B_n(x) = \prod_{r=1}^{n} \left[x + 4 \cos^2 \left(\frac{r}{n+1}\right) \cdot \frac{\pi}{2} \right]$$

holds. Does this result hold good for all n? Is it possible to find a similar result for $b_n(x)$?

SOLUTIONS

FROM BEST SET OF K TO BEST SET OF K+1 ?

H-42 *Proposed by J. D. E. Konhauser, State College, Pa.*

A set of nine integers having the property that no two pairs have the same sum is the set consisting of the nine consecutive Fibonacci numbers, 1, 2, 3, 5, 8, 13, 21, 34, 55 with total sum 142. Starting with 1, and annexing at each step the smallest positive integer which produces a set with the stated property yields the set 1, 2, 3, 5, 8, 13, 21, 30, 39 with sum 122. Is this the best result? Can a set with lower total sum be found?

Partial solution by the proposer.

Partial answer. The set 1, 2, 4, 5, 9, 14, 20, 26, 35 has total sum 116. For eight numbers the best set appears to be 1, 2, 3, 5, 9, 15, 20, 25 with sum 80. Annexing the lowest possible integer to extend the set to nine members requires annexing 38 which produces a set with sum 118. It is not clear (to me, at least) how to progress from a best set of k integers to a best set for k + 1 integers.

Comments by Murray Berg, Oakland, California

The set given above in the partial solution is invalid since $1+5 = 4+2 = 6$ and the problem asks for distinct sums for different pairs.

Comments by the Editor

An apparent solution summing to 118 was received but was discarded since the sum was larger than the partial solution given above. Please resubmit if you read this.

AT LAST A SOLUTION

H-26 *Proposed by L. Carlitz, Duke University (corrected)*

Let $R_k = (b_{rs})$, where $b_{rs} = \binom{r-1}{k+1-2}$ $(r, s = 1, 2, \ldots. k+1)$. Then show

$$R_k^n = \left(\sum_{j=1}^{s} \binom{r-1}{j-1} \binom{k+1-r}{s-j} F_{n-1}^{k+1-r-s+j} F_n^{r+s-2j} F_{n+1}^{j-1} \right)$$

Letting $R_k^n = (a_{rs})$, we evaluate a_{rs} by extending the proposer's method of solving B-16 (Fibonacci Quarterly, Vol. 2, No. 2, pp. 155-157). Using Carlitz's notation, we may easily show by induction that the transformation

$$T_1 : \begin{cases} x' = y \\ y' = x + y \end{cases}$$

induces the transformation

$$T_k: \begin{cases} x'^k = \binom{0}{0}y^k \\ x'^{k-1}y' = \binom{1}{1}xy^{k-1} + \binom{1}{0}y^k \\ \vdots \\ x'y'^{k-1} = \binom{k-1}{k-1}x^{k-1}y + \dots + \binom{k-1}{1}xy^{k-1} + \binom{k-1}{0}y^k \\ y'^k = \binom{k}{k}x^k + \binom{k}{k-1}x^{k-1}y + \dots + \binom{k}{1}xy^{k-1} + \binom{k}{0}y^k \end{cases}$$

Carlitz also showed that the T_1^n is given by

$$(1) \qquad T_1^n : \begin{cases} x^{(n)} = F_{n-1}x + F_n y \\ y^{(n)} = F_n x + F_{n+1} y \end{cases}$$

so that T_k^n induces the transformation

$$(2) \quad T_k^n: \left\{ (x^{(n)})^{k-r+1}(y^{(n)})^{r-1} = \sum_{j=1}^{k+1} a_{rs}x^{k+1-s}y^{s-1} \quad (r = 1, 2, \dots, k+1), \right.$$

We note here a misprint in the B-16 solution: the last transformation should begin with T_2^n instead of T_1^n. To evaluate a_{rs}, we substitute (1) into (2) to obtain

$$\sum_{s=1}^{k+1} a_{rs}x^{k+1-s}y^{s-1} = (F_{n-1}x + F_n y)^{k+1-r}(F_n x + F_{n+1}y)^{r-1}$$

$$= \sum_{i=0}^{k+1-r} \binom{k+1-r}{i} F_{n-1}^{k-1-r-i}F_n^i x^{k+1-r-i}y^i$$

$$\times \sum_{j=0}^{r-1} \binom{r-1}{j} F_n^{r-1-j}F_{n+1}^j x^{r-1-j}y^j$$

$$= \sum_{i=0}^{k+1-r} \sum_{j=0}^{r-1} \binom{k+1-r}{i}\binom{r-1}{j} F_{n-1}^{k+1-r-i}F_n^{r-1+i-j}F_{n+1}^j x^{k-i-j}y^{i+j} .$$

We equate coefficients of $x^{k+1-s}y^{s-1}$, summing all terms of the last sum with $i+j = s-1$, and since $j \leq s-1$ we find

$$a_{rs} = \sum_{j=0}^{s-1} \binom{k+1-r}{s-1-j}\binom{r-1}{j-1} F_{n-1}^{k+2-r-s+j} F_n^{r+s-2-2j} F_{n+1}^{j}$$

$$= \sum_{j=1}^{s} \binom{k+1-r}{s-j}\binom{r-1}{j-1} F_{n-1}^{k+1-r-s+j} F_n^{r+s-2j} F_{n+1}^{j-1} .$$

ANOTHER LATE ONE

H-38 *Proposed by R. G. Buschman, SUNY, Buffalo, N. Y.*

(See Fibonacci Numbers, Chebyshev Polynomials, Generalizations and Difference Equations; Vol. 1, No. 4, Dec. 1963, pp. 1-7.) Show

$$(u_{n+r} + (-b)^r u_{n-r})/u_n = \lambda_r .$$

Solution by Douglas Lind

Let $z_1 \neq z_2$ be the roots of $z^2 - az - b = 0$, and note $a = z_1 + z_2$, $-b = z_1 z_2$. We recall from the article that

$$u_n = \left\{ (u_1 - z_1 u_0) z_2^n - (u_1 - z_2 u_0) z_1^n \right\} / (z_2 - z_1)$$

and

$$\lambda_n = \left\{ (a - 2z_1) z_2^n - (a - 2z_2) z_1^n \right\} / (z_2 - z_1) .$$

Now

$$\lambda_n = z_2^n + z_1^n$$

since $a - 2z_1 = z_2 - z_1 = -(a - 2z_2)$, so that

$$u_n \lambda_r = \left\{ (u_1 - z_1 u_0) z_2^{n+r} - (u_1 - z_2 u_0) z_1^{n+r} + (-b)^r (u_1 - z_1 u_0) z_2^{n-r} - (-b)^r (u_1 - z_2 u_0) z_1^{n-r} \right\} / (z_2 - z_1)$$

$$= u_{n+r} + (-b)^r u_{n-r}$$

the desired result.

Also solved by Clyde Bridger and the proposer.