A GENERATING FUNCTION FOR FIBONACCI NUMBERS

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Since interesting identities for certain number theoretic functions can be derived from their generating functions, in particular generating functions for Dirichlet series, the following problem seemed to be of interest.

Problem: Find a generating function G which yields the Fibonacci numbers in the coefficients of a Dirichlet series.

First we note that we must write the series in the form

$$(1) \qquad G(s) = \sum_{n=1}^{\infty} f_n c_n n^{-s},$$

since the series diverges for $c_n \equiv 1$, the f_n's increase too rapidly. Part of the goal is, as a result, to find a simple expression to use for c_n.

One attempt at the solution proceeds as follows. Consider the more general difference equation,

$$(2) \qquad u_0,\ u_1,\ u_{n+1} = au_n + bu_{n-1} \quad (n \geqq 1),$$

from which we can write

$$u_n = \left[z_2^n(u_1 - z_1u_0) - z_1^n(u_1 - z_2u_0)\right]/(z_2 - z_1)$$

with $z_1z_2 = -b$, $z_1+z_2 = a$, $z_1 \neq z_2$. Substituting into the Dirichlet series we have

$$(3) \qquad \sum_{n=1}^{\infty} u_n c_n n^{-s} = A(z_1) \sum_{n=1}^{\infty} c_n z_2^n n^{-s} + A(z_2) \sum_{n=1}^{\infty} c_n z_1^n n^{-s}$$

where the function A is defined by

$$A(z_1) = (u_0z_1^2 - u_1z_1)/(z_1^2+b) = (u_1 - z_1u_0)/(z_2 - z_1).$$

Since c_n must be chosen to guarantee the convergence of the series in (3), it is convenient to select $c_n = c$ and then $|cz_2| < 1$, $|cz_1| < 1$. Hence equation (3) can be written

$$(4) \qquad \sum_{n=1}^{\infty} u_n c^n n^{-s} = A(z_1)\, F(az_2, s) + A(z_2)\, F(az_1, s) \quad ,$$

where $F(z, s)$ is a function discussed by Truesdell [2]. Further

$$F(z, s) = \sum_{n=1}^{\infty} z^n n^{-s} = z \; \Phi(z, s, 1)\,,$$

where Φ denotes the Lerch Zeta-function - some of the properties of which are known [1:1.11]. This allows the result to be expressed in various forms.

The difference equation (2) can be rewritten for $c^n u_n = v_n$ in the form

$$v_0,\ v_1,\ v_{n+1} = acv_n + bc^2 v_{n-1} \quad (n \geqq 1)\ .$$

For the Fibonacci case it is convenient to set $c = 1/2$, so that the generating function for $2^{-n}f_n$, that is

$$G(s) = \sum_{n=1}^{\infty} (2^{-n}f_n)\, n^{-s} \quad ,$$

can be written in the form

$$(5) \qquad G(s) = (2/\sqrt{5}) \left\{ F\left[(1+\sqrt{5})/4, s\right] - F\left[(1-\sqrt{5})/4, s\right] \right\} \ .$$

To make efficient use of this generating function one needs to have available identities involving the function $F(z, s)$, especially such identities as involve products. Analogous to the ζ-function, an infinite product expansion for $F(z, s)$ in terms of s, with fixed z, might be helpful.

REFERENCES

1. A. Erdélyi, et al., High Transcendental Functions, Vol. 1, McGraw-Hill, New York, 1953.
2. C. A. Truesdell, "On a Function which occurs in the Theory of the Structure of Polymers," Ann. of Math. 46(1945), pp. 144-151.

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