ON GLAISHER'S INFINITE SUMS INVOLVING THE INVERSE TANGENT FUNCTION

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1. Introduction

In 1878, J. W. L. Glaisher [1] derived a number of results about certain infinite sums involving the inverse tangent function; in particular, he showed for complex θ $(0 < |\theta| < \infty)$, that

$$\sum_{n=1}^{\infty} \arctan \frac{2\theta^2}{n^2} = \frac{\pi}{4} - \arctan\left(\frac{\tanh \pi\theta}{\tan \pi\theta}\right). \tag{1}$$

This equation appears again in 1908 as an exercise in T. J. I'a. Bromwich's book [2, p. 259]. Generalizations of (1) are found in [3], [4, p. 276], and [5, p. 749].

Letting $\theta \to 1-$ in (1), Glaisher also obtained the elegant result:

$$\sum_{n=1}^{\infty} \arctan \frac{2}{n^2} = \frac{3}{4}\pi. \tag{2}$$

A very simple derivation of (2) and a history of this series appeared recently in [6].

It is easy to see that the two members of (1) may differ by an integer multiple of π; this pathology occurs often in many results of this type, since the inverse tangent function is a multiple-valued function. Hence, if we use only the principal value of the inverse tangent function, we must write (1) in the form

$$\sum_{n=1}^{\infty} \arctan \frac{2\theta^2}{n^2} = \left(\frac{1}{4} + m\right)\pi - \arctan\left(\frac{\tanh \pi\theta}{\tan \pi\theta}\right) \tag{3}$$

for some $m \in \mathbb{Z} \equiv \{0, \pm 1, \pm 2, \ldots\}$.

In this paper we shall derive computationally more useful results than (3); our results will yield some interesting corollaries not available heretofore. Indeed we shall show, for complex θ $(0 < |\theta| < \infty)$, that

$$\sum_{n=1}^{\infty} \arctan \frac{2\theta^2}{n^2} = \left(\theta - \frac{1}{4}\right)\pi - \arctan\left(\frac{\sin 2\pi\theta}{\cos 2\pi\theta - \exp 2\pi\theta}\right) \tag{4}$$

where, here and in what follows, the principal value of the inverse tangent function is assumed. We shall also show that (3) and (4) are, in fact, equivalent. We shall then give (in Section 5) some generalizations of (4). Finally, in Section 6, we deduce some interesting particular cases of one of the general summation formulas which we obtain in Section 5.

2. Derivation of the Summation Formula (4)

To derive (4), we shall use the Euler-Maclaurin summation formula ([7, p. 27]; see also [8, p. 521])

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)dx + \frac{1}{2}f(0) + \frac{1}{2}f(n) + \int_0^n P(x)f'(x)dx$$

where $P(x)$, for real x, is a saw-tooth function: $P(x) = x - [x] - 1/2$. Letting $f(x) = \arctan(2\theta^2/x^2)$ and $n \to \infty$, we obtain

$$\sum_{k=0}^{\infty} \arctan \frac{2\theta^2}{k^2} = \int_0^{\infty} \arctan \frac{2\theta^2}{x^2} dx + \frac{\pi}{4} - 4\theta^2 \int_0^{\infty} P(x) \frac{x dx}{4\theta^4 + x^4}.$$

Assuming $0 < \theta < \infty$ and making simple transformations in the integrals, we have

$$(5) \qquad \sum_{k=1}^{\infty} \arctan \frac{2\theta^2}{k^2} = -\frac{\pi}{4} + \theta \int_0^{\infty} \arctan \frac{2}{x^2} dx - 2\int_0^{\infty} P(\theta\sqrt{2}x) \frac{x dx}{1 + x^4}.$$

The first integral on the right side of (5) can be evaluated in a number of ways or by using tables of integrals (cf. [5] and [9]). We omit the details and give the result:

$$(6) \qquad \int_0^{\infty} \arctan(2/x^2) dx = \pi.$$

The saw-tooth function $P(x)$ is a sectionally (piecewise) smooth periodic function with unit period. It can be represented by a Fourier series which is given by

$$(7) \qquad P(x) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi k x).$$

The series given in (7) converges uniformly in every closed interval where $P(x)$ is continuous. The saw-tooth function and its Fourier series representation are discussed in detail, for example, in [10, pp. 107-24].

To evaluate the second integral in (5), we use (7) and interchange the sum and integral, thus giving:

$$(8) \qquad \int_0^{\infty} P(\theta\sqrt{2}x) \frac{x dx}{1 + x^4} = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\infty} \sin(2\sqrt{2}\theta\pi k x) \frac{x dx}{1 + x^4}.$$

Using [9, p. 408, Sec. 3.727, Eq. (4)], we find that

$$(9) \qquad \int_0^{\infty} \sin(2\sqrt{2}\theta\pi k x) \frac{x dx}{1 + x^4} = \frac{\pi}{2} \exp(-2\theta\pi k) \sin(2\theta\pi k).$$

Hence, from (5), (6), (8), and (9), we obtain

$$\sum_{k=1}^{\infty} \arctan \frac{2\theta^2}{k^2} = \left(\theta - \frac{1}{4}\right)\pi + \sum_{k=1}^{\infty} \frac{1}{k} \exp(-2\theta\pi k) \sin(2\theta\pi k);$$

and now, using [5, p. 740, Eq. (5)], we can write the sum on the right in closed form, thus giving (4), provided that $0 < \theta < \infty$.

It can easily be shown that the right member of (4) is indeed an even function of θ and that, as θ approaches zero, it vanishes. Hence, (4) is valid for real θ and (by appealing to the principle of analytic continuation) it is valid for complex θ. This evidently completes the derivation of the summation formula (4).

3. Equivalence of the Sums (3) and (4)

Defining

$$\xi(x) \equiv \frac{\sin 2x}{\cos 2x - \exp 2x},$$

we note the easily verified identity

$$\frac{\tan x}{\tanh x} = \frac{\tan x - \xi(x)}{1 + \xi(x) \tan x}.$$

Since $\tan x = \tan(x - m\pi)$, for all $m \in \mathbb{Z}$, this gives

$$\frac{\tan x}{\tanh x} = \frac{\tan(x - m\pi) - \xi(x)}{1 + \xi(x)\tan(x - m\pi)}.$$

Taking the inverse tangent of both members of this equation and observing that

$$\arctan u - \arctan v = \arctan((u - v)/(1 + uv)),$$

we obtain

$$\arctan\left(\frac{\tan x}{\tanh x}\right) = (x - m\pi) - \arctan \xi(x).$$

Now, using $\arctan x = \pi/2 - \arctan 1/x$, we deduce from this the identity

$$(10) \qquad \frac{\pi}{2} - \arctan\left(\frac{\tanh x}{\tan x}\right) + m\pi = x - \arctan\left(\frac{\sin 2x}{\cos 2x - \exp 2x}\right)$$

for some $m \in \mathbb{Z}$. Replacing x by $\theta\pi$, (10) shows that the results in (3) and (4) are indeed equivalent.

4. Special Cases of Equation (4)

In (4), if we set $\theta = k$ and $\theta = k/2$ ($k = 1, 2, 3, \ldots$), we obtain

$$(11) \qquad \sum_{n=1}^{\infty} \arctan \frac{2k^2}{n^2} = \left(k - \frac{1}{4}\right)\pi$$

and

$$(12) \qquad \sum_{n=1}^{\infty} \arctan \frac{k^2}{2n^2} = \left(k - \frac{1}{2}\right)\frac{\pi}{2},$$

respectively; now, splitting the sum in (11) into even and odd terms, and using (12), we deduce also that

$$(13) \qquad \sum_{n=0}^{\infty} \arctan \frac{2k^2}{(2n + 1)^2} = \frac{\pi}{2}k.$$

Equation (2) follows from (11) when $k = 1$. Equations (12) and (13) were also derived by Glaisher for $k = 1$. Ramanujan (circa 1903) derived (11), (12), and (13) for $k = 1$ [11, Ch. 2].

5. Generalizations of the Summation Formula (4)

Letting $f(x) = \arctan(z^{2n}/x^{2n})$ in the Euler-Maclaurin summation formula (cited already in Section 2), but now using [9, p. 608, Sec. 4.532, Eq. (2)] and [5, p. 396, Eq. (2)] to compute the two integrals, in basically the same way as (4) was obtained, we can derive the result

$$(14) \qquad \sum_{k=1}^{\infty} \arctan \frac{z^{2n}}{k^{2n}} = \left(z \sec \frac{\pi}{4n} - \frac{1}{2}\right)\frac{\pi}{2} + \sum_{k=1}^{n} (-1)^k \arctan\left(\frac{\sin \xi}{\cos \xi - \exp \eta}\right)$$

$$(0 < |z| < \infty;\ n = 1, 2, 3, \ldots),$$

where

$$\xi = 2\pi z \cos \frac{2k - 1}{4n}\pi, \quad \eta = 2\pi z \sin \frac{2k - 1}{4n}\pi.$$

For $n = 1$ and $z = \sqrt{2}\theta$, (14) reduces to (4). For $n = 2$, setting $\alpha = \pi x \cos \pi/8$ and $\beta = \pi x \sin \pi/8$, we get

$$(15) \qquad \sum_{k=1}^{\infty} \arctan \frac{x^4}{k^4} = \left[\alpha - \arctan\left(\frac{\sin 2\alpha}{\cos 2\alpha - \exp 2\beta}\right)\right] - \left[\beta - \arctan\left(\frac{\sin 2\beta}{\cos 2\beta - \exp 2\alpha}\right)\right] - \frac{\pi}{4}.$$

Glaisher [1] obtained, modulo an integer multiple of π, that

$$(16)\qquad \sum_{k=1}^{\infty} \arctan \frac{x^4}{k^4}$$
$$= \arctan\left(\frac{\tan\alpha \tanh\alpha - \tan\beta \tanh\beta - \tan\alpha \tan\beta - \tanh\alpha \tanh\beta}{\tan\alpha \tanh\alpha - \tan\beta \tanh\beta + \tan\alpha \tan\beta + \tanh\alpha \tanh\beta}\right).$$

Hence, the difference of the right members of (15) and (16) is an integer multiple of π.

By splitting the left member of (14) into even and odd terms, we easily find that

$$(17)\qquad \sum_{k=0}^{\infty} \arctan \frac{z^{2n}}{(2k+1)^{2n}} = \frac{\pi z}{4} \sec \frac{\pi}{4n}$$
$$+ \sum_{k=1}^{n} (-1)^k \left[\arctan\left(\frac{\sin\xi}{\cos\xi - \exp\eta}\right) - \arctan\left(\frac{\sin\xi/2}{\cos\xi/2 - \exp\eta/2}\right)\right]$$
$$(n = 1, 2, 3, \ldots).$$

Glaisher [1] also obtained results, modulo an integer multiple of π, for the left member of (17) in the special cases when $n = 1$ and $n = 2$.

We note here that, in general, when an infinite sum of arctangent functions is given modulo an integer multiple of π, the Euler-Maclaurin summation formula appears to be helpful in attempting to derive computationally more useful results.

By using (14) and (17), we have, in addition,

$$\sum_{k=1}^{\infty} (-1)^{k+1} \arctan \frac{z^{2n}}{k^{2n}} = \frac{\pi}{4}$$
$$+ \sum_{k=1}^{n} (-1)^k \left[\arctan\left(\frac{\sin\xi}{\cos\xi - \exp\eta}\right) - 2\arctan\left(\frac{\sin\xi/2}{\cos\xi/2 - \exp\eta/2}\right)\right].$$

In particular, letting $n = 1$ and $z = \sqrt{2}\theta$, we get

$$(18)\qquad \sum_{k=1}^{\infty} (-1)^{k+1} \arctan \frac{2\theta^2}{k^2} = \frac{\pi}{4} - \arctan\left(\frac{\sin 2\pi\theta}{\cos 2\pi\theta - \exp 2\pi\theta}\right)$$
$$+ 2\arctan\left(\frac{\sin\pi\theta}{\cos\pi\theta - \exp\pi\theta}\right).$$

By using [4, p. 277, Eq. (42.1.10)], (18) may be written equivalently as

$$(19)\qquad \sum_{k=1}^{\infty} (-1)^{k+1} \arctan \frac{2\theta^2}{k^2} = \arctan\left(\frac{\sinh\pi\theta}{\sin\pi\theta}\right) - \frac{\pi}{4}.$$

6. A Special Case of Formula (18)

In (18) or (19), if we set $\theta = \ell$ ($\ell = 1, 2, 3, \ldots$), we deduce the intesting result:

$$(19)\qquad \sum_{k=1}^{\infty} (-1)^{k+1} \arctan \frac{2\ell^2}{k^2} = \frac{\pi}{4} \quad (\ell = 1, 2, 3, \ldots),$$

from which it easily follows that

$$\sum_{k=1}^{\infty} (-1)^{k+1} \arctan\left[\frac{2(\ell^2 - m^2)k^2}{k^4 + 4\ell^2 m^2}\right] = 0 \quad (m = 1, 2, 3, \ldots)$$

and

$$\sum_{k=1}^{\infty} (-1)^{k+1} \arctan\left[\frac{2(\ell^2 + m^2)k^2}{k^4 - 4\ell^2 m^2}\right] = \frac{\pi}{2} \quad (m = 1, 2, 3, \ldots),$$

ℓ being a positive integer.

Equation (19) apparently was first derived by Ramanujan for the special case $\ell = 1$ [11, Ch. 2] and it is also derived for $\ell = 1$ by Wheelon [12, p. 46].

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Author and Title Index for *The Fibonacci Quarterly*

Currently, Dr. Charles K. Cook of the University of South Carolina at Sumter is working on an AUTHOR index, TITLE index and PROBLEM index for *The Fibonacci Quarterly.* In fact, these three indices are already completed. We hope to publish these indices in 1993 which is the 30th anniversary of *The Fibonacci Quarterly.* Dr. Cook and I feel that it would be very helpful if the publication of the indices also had AMS classification numbers for all articles published in *The Fibonacci Quarterly.* We would deeply appreciate it if all authors of articles published in *The Fibonacci Quarterly* would take a few minutes of their time and send a list of articles with primary and secondary classification numbers to

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At their summer meeting, the board of directors voted to publish the indices on a 3.5-inch high density disk. The price will be \$40.00 to nonmembers and \$20.00 to members plus postage. Disks will be available for use on the MacIntosh or any IBM compatible machine using Word Perfect, Word, First Choice or any of a number of other word processors. More on this will appear in the February 1993 issue.

Gerald E. Bergum, **Editor**