ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

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Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-736 ***Proposed by Herta T. Freitag, Roanoke, VA***

Prove that $(2L_n + L_{n-3})/5$ is a Fibonacci number for all n.

B-737 ***Proposed by Herta T. Freitag, Roanoke, VA***

A right triangle, one of whose legs is twice as long as the other leg, has a hypotenuse that is one unit longer than the longer leg. Let r be the inradius of this triangle (radius of inscribed circle) and let r_a, r_b, r_c be the exradii (radii of circles outside the triangle that are tangent to all three sides).

Express r, r_a, r_b, and r_c in terms of the golden ratio, α.

B-738 ***Proposed by Daniel C. Fielder & Ceceil O. Alford, Georgia Institute of Technology, Atlanta, GA***

Find a polynomial $f(w,x,y,z)$ such that

$$f(L_n, L_{n+1}, L_{n+2}, L_{n+3}) = 25 f(F_n, F_{n+1}, F_{n+2}, F_{n+3})$$

is an identity.

B-739 ***Proposed by Ralph Thomas, University of Chicago, Dundee, IL***

Let $S = \left\{ \frac{F_i}{F_j} \,\middle|\, i \geq 0, j > 0 \right\}$. Is S dense in the set of nonnegative real numbers?

B-740 ***Proposed by Thomas Martin, Phoenix, AZ***

Find all positive integers x such that 10 is the smallest integer, n, such that $n!$ is divisible by x.

B-741 ***Proposed by Jayantibhai M. Patel, Bhavan's R. A. College of Science, Gujarat, India***

Prove that $F_{n+8}^4 + 331F_{n+4}^4 + F_n^4$ is always divisible by 54.

SOLUTIONS

Coefficients of a Maclaurin Series

B-709 ***Proposed by Alejando Necochea, Pan American University, Edinburg, TX***

Express

$$\frac{1}{n!}\frac{d^n}{dt^n}\left[\frac{t}{1-t-t^2}\right]_{t=0}$$

in terms of Fibonacci numbers.

Solution by Douglas E. Iannucci, Riverside, RI

Since $t/(1-t-t^2)$ is the generating function for the Fibonacci sequence, we have

$$\sum_{k=0}^{\infty} F_k t^k = \frac{t}{1-t-t^2}$$

(see [2], pp. 52-53). Thus

$$\frac{1}{n!}\frac{d^n}{dt^n}\left[\frac{t}{1-t-t^2}\right]_{t=0} = \frac{1}{n!}\frac{d^n}{dt^n}\left[\sum_{k=0}^{\infty} F_k t^k\right]_{t=0}$$

$$= \frac{1}{n!}\left[\sum_{k=n}^{\infty} k(k-1)(k-2)\cdots(k-n+1)F_k t^{k-n}\right]_{t=0} = \left[\sum_{k=n}^{\infty}\binom{k}{n}F_k t^{k-n}\right]_{t=0} = F_n.$$

Several solvers blithely proceeded to differentiate the power series for $t/(1-t-t^2)$ term by term (as above) without justifying that this produces correct results. Several solvers quoted Taylor's Theorem, but this did not convince me. The procedure is valid by the following ([1], p. 26):

Theorem: In the interior of its circle of convergence, a power series may be differentiated term by term. The derived series converges and represents the derivative of the sum of the original power series. Furthermore, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence $R > 0$, then $f(z)$ has derivatives of all orders; and for $|z| < R$, we have

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n z^{n-k}.$$

References:

1. Einar Hille. *Analytic Function Theory.* Vol. 1. New York: Blaisdell, 1959.
2. S. Vajda. *Fibonacci and Lucas Numbers, and the Golden Section—Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Also solved by Glenn Bookhout, Wray Brady, Paul S. Bruckman, Joseph E. Chance, Russell Euler, C. Georghiou, Russell Jay Hendel, Joseph J. Kostal, Igor Ol. Popov, Bob Prielipp, Don Redmond, H.-J. Seiffert, Sahib Singh, Ralph Thomas, and the proposer.

Pell-Lucas Congruences

B-710 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let P_n be the n^{th} Pell number, defined by $P_0 = 0, P_1 = 1, P_{n+2} = 2P_{n+1} + P_n$, for $n \geq 0$. Prove that

(a) $P_{3n+1} \equiv L_{3n+1} \pmod 5$,
(b) $P_{3n+2} \equiv -L_{3n+2} \pmod 5$.
(c) Find similar congruences relating Pell numbers and Fibonacci numbers.

Solution by Paul S. Bruckman, Edmonds, WA

We may readily form the following short table of P_n and L_n modulo 5:

n	$P_n \pmod 5$	$L_n \pmod 5$
0	0	2
1	1	1
2	2	−2
3	0	−1
4	2	2
5	−1	1
6	0	−2
7	−1	−1
8	−2	2
9	0	1
10	−2	−2
11	1	−1
12	0	2
13	1	1

Inspection of the foregoing table shows that P_n repeats every 12 terms and L_n repeats every 4 terms. Thus, to discover the relations between the P_n's and L_n's (mod 5), it suffices to consider the first 12 terms of the sequences involved. Parts (a) and (b) of the problem then follow immediately by inspecting this table and confirming the congruences.

To find relations between the P_n's and F_n's, we form a similar array tabulating P_n and F_n modulo 5. We find that F_n repeats every 20 terms. Thus, it suffices to consider the first 60 terms of the sequences involved (since lcm[12, 20] = 60). We omit the tabulation, but the 60-line table is straightforward to create. Inspecting that table, we find

$$\begin{aligned} P_{15n+a} &\equiv F_{15n+a} \pmod 5 && \text{for } a \in \{-1, 0, 1\} \\ P_{15n+a} &\equiv -F_{15n+a} \pmod 5 && \text{for } a \in \{-4, 0, 4\} \\ 2P_{15n+a} &\equiv -F_{15n+a} \pmod 5 && \text{for } a \in \{-2, 0, 2\} \\ 2P_{15n+a} &\equiv F_{15n+a} \pmod 5 && \text{for } a \in \{-7, 0, 7\}. \end{aligned}$$

This last set of congruences repeat every 15 entries (rather than every 60) because of the fact that $P_{n+15} \equiv 2P_n \pmod 5$ and $F_{n+15} \equiv 2F_n \pmod 5$.

Georghiou, Prielipp, Somer, and Thomas found the following congruences modulo 3:

$$\begin{aligned} P_{2n} &\equiv -F_{2n} \pmod 3 \\ P_{2n+1} &\equiv F_{2n+1} \pmod 3 \end{aligned}$$

which can also be expressed as $P_n \equiv (-1)^{n+1} F_n \pmod 3$.

Also solved by Charles Ashbacher (parts a and b), Herta T. Freitag, C. Georghiou, Russell Jay Hendel, Joseph J. Kostal, Bob Prielipp, Stanley Rabinowitz, Lawrence Somer, Ralph Thomas, and the proposer.

Cosh, What a Product!

B-711 ***Proposed by Mihály Bencze, Sacele, Romania***

Let r be a natural number. Find a closed form expression for

$$\prod_{k=1}^{\infty}\left(1-\frac{L_{4r}}{k^4}+\frac{1}{k^8}\right).$$

Solution by H.-J. Seiffert, Berlin, Germany

It is known that $\sin \pi z$ and $\sinh \pi z$ have the following product expansions:

$$\sin \pi z = \pi z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right) \tag{1}$$

and

$$\sinh \pi z = \pi z\prod_{k=1}^{\infty}\left(1+\frac{z^2}{k^2}\right) \tag{2}$$

which are valid for all complex z (see [1], series 4.3.89 and 4.5.68; [2], p. 350; [3], p. 37, section 1.431; or [4], series 1016 and 1078).

Thus,

$$\begin{aligned}
P &= \prod_{k=1}^{\infty}\left(1-\frac{L_{4r}}{k^4}+\frac{1}{k^8}\right) = \prod_{k=1}^{\infty}\left(1-\frac{\alpha^{4r}}{k^4}\right)\left(1-\frac{\beta^{4r}}{k^4}\right)\\
&= \prod_{k=1}^{\infty}\left(1-\frac{\alpha^{2r}}{k^2}\right)\left(1+\frac{\alpha^{2r}}{k^2}\right)\left(1-\frac{\beta^{2r}}{k^2}\right)\left(1+\frac{\beta^{2r}}{k^2}\right)\\
&= \prod_{k=1}^{\infty}\left(1-\frac{\alpha^{2r}}{k^2}\right)\prod_{k=1}^{\infty}\left(1+\frac{\alpha^{2r}}{k^2}\right)\prod_{k=1}^{\infty}\left(1-\frac{\beta^{2r}}{k^2}\right)\prod_{k=1}^{\infty}\left(1+\frac{\beta^{2r}}{k^2}\right)\\
&= \frac{\sin(\pi\alpha^r)}{\pi\alpha^r}\,\frac{\sinh(\pi\alpha^r)}{\pi\alpha^r}\,\frac{\sin(\pi\beta^r)}{\pi\beta^r}\,\frac{\sinh(\pi\beta^r)}{\pi\beta^r}\\
&= \frac{1}{\pi^4}\sin(\pi\alpha^r)\sin(\pi\beta^r)\sinh(\pi\alpha^r)\sinh(\pi\beta^r).
\end{aligned}$$

Editorial note: *The step wherein we pass to a product of four infinite products needs some justification. The infinite product* $\prod_{n=1}^{\infty}(1+x_n)$ *is said to be absolutely convergent if the series* $\sum_{n=1}^{\infty}|x_n|$ *is convergent (see [2], p. 159). All the infinite products used above are absolutely convergent. It is known that the factors of an absolutely convergent infinite product may be rearranged arbitrarily without affecting the convergence of the product (see [5], p. 530). Thus, we are permitted to equate* $\prod a_k b_k$ *and* $\prod a_k \prod b_k$*, which justifies the above procedures.*

The result can be simplified further. Using the formulas

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)] \text{ and } \sinh x \sinh y = \frac{1}{2}[\cosh(x+y) - \cosh(x-y)]$$

(see [1], formula 4.5.38), we find that

$$P = \frac{1}{4\pi^4}\left[\cos\left(\pi F_r\sqrt{5}\right) - \cos\left(\pi L_r\right)\right]\left[\cosh\left(\pi L_r\right) - \cosh\left(\pi F_r\sqrt{5}\right)\right].$$

References:

1. Milton Abramowitz & Irene A. Stegun. *Handbook of Mathematical Functions.* Washington, DC: National Bureau of Standards, 1964.
2. G. Chrystal. *Textbook of Algebra.* Part 2. New York: Dover, 1961.
3. I. S. Gradshteyn & I. M. Ryzhik. *Tables of Integrals, Series and Products.* San Diego, CA: Academic Press, 1980.
4. L. B. W. Jolley. *Summation of Series.* 2nd rev. ed. New York: Dover, 1961.
5. Edgar M. E. Wermuth. "Some Elementary Properties of Infinite Products." *Amer. Math. Monthly* **99** (1992):530-37.

Also solved by Paul S. Bruckman, C. Georghiou, Igor Ol. Popov, and the proposer.

Another Lucas Number

B-712 ***Proposed by Herta T. Freitag, Roanoke, VA***

Prove that for all positive integers n, $\alpha(\sqrt{5}\alpha^n - L_{n+1})$ is a Lucas number.

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

Since $\sqrt{5} = \alpha - \beta$ and $\alpha\beta = -1$, we have

$$\begin{aligned}\alpha(\sqrt{5}\alpha^n - L_{n+1}) &= \alpha[(\alpha-\beta)\alpha^n - (\alpha^{n+1} + \beta^{n+1})] \\ &= \alpha^{n+2} - \beta\alpha^{n+1} - \alpha^{n+2} - \alpha\beta^{n+1} \\ &= \alpha^n + \beta^n = L_n.\end{aligned}$$

Most solutions were similar. Redmond found an analog for Fibonacci numbers: $\alpha(\alpha^n - F_{n+1}) = F_n$. *Haukkanen found this too, as well as* $-\beta(\sqrt{5}\beta^n + L_{n+1}) = L_n$ and $\beta(\beta^n - F_{n+1}) = F_n$. *Redmond also generalized to arbitrary second-order linear recurrences.*

Also solved by Richard André-Jeannin, Charles Ashbacher, Mohammad K. Azarian, M. A. Ballieu, Seung-Jin Bang, Glenn Bookhout, Scott H. Brown, Paul S. Bruckman, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, Jane Friedman, Pentti Haukkenen, Russell Jay Hendel, Carl Libis, Graham Lord, Dorka Ol. Popova, Bob Prielipp, Don Redmond, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Ralph Thomas, and the proposer.

Complex Pythagorean Triple

B-713 ***Proposed by Herta T. Freitag, Roanoke, VA***

(a) Let S be a set of three consecutive Fibonacci numbers. In a Pythagorean triple, (a, b, c), a is the product of the elements in S; b is the product of two Fibonacci numbers (both larger than

1), one of them occurring in S; and c is the sum of the squares of two members of S. Determine the Pythagorean triple and prove that the area of the corresponding Pythagorean triangle is the product of four consecutive Fibonacci numbers.

(b) Same problem as part (a) except that Fibonacci numbers are replaced by Lucas numbers.

Solution by Paul S. Bruckman, Edmonds, WA

Part (a): Let $S=(F_{n-1},F_n,F_{n+1})$. Since we require $a=F_{n-1}F_nF_{n+1}$ to be the side of a triangle, we must have $n\geq 2$. Also, if $n=2$, then $a=2$, which cannot be the side of a Pythagorean triangle. Thus, $n\geq 3$. Since the sequence $(F_n)_{n=2}^{\infty}$ is increasing, the hypothesis implies that $c\leq F_n^2+F_{n+1}^2=F_{2n+1}$. Also, we must have $c>a$. Thus, we are to have $F_{n-1}F_nF_{n+1}<F_{2n+1}$. This inequality can be satisfied only for a finite number of n. In fact, it holds only for $n\leq 4$. Thus, the only possible solutions are generated by $n=3$ or $n=4$.

If $n=3$, we obtain the value $a=F_4F_3F_2=3\cdot 2\cdot 1=6$. Since $c>a$, we must have $c=3^2+2^2=13$ or $c=3^2+1^2=10$. However, $13^2-6^2=133$, which is not a perfect square; so we reject that possibility. Since $10^2-6^2=8^2$, we try $b=8$. However, $8=8\cdot 1=F_6F_2$, which is the only factorization of 8 into two factors that are Fibonacci numbers. Since $F_2=1$, we must also reject this possibility.

If $n=4$, we obtain $a=F_5F_4F_3=5\cdot 3\cdot 2=30$. The only possible value for c is $F_5^2+F_4^2=5^2+3^2=34$. This yields $b^2=34^2-30^2=16^2$, so $b=16$. We can factor 16 as a product of two Fibonacci numbers in only one way; namely, $16=8\cdot 2=F_6F_3$, and both factors are larger than 1. Moreover, F_3 divides a. Therefore, this is a valid solution and is, indeed, the only solution:

$$(a,b,c)=(F_5F_4F_3,F_6F_3,F_5^2+F_4^2)=(5\cdot 3\cdot 2,8\cdot 2,5^2+3^2)=(30,16,34).$$

For this unique solution, the area of the triangle formed by sides of length a, b, and c is equal to $\frac{1}{2}\cdot 30\cdot 16=240=2\cdot 3\cdot 5\cdot 8=F_3F_4F_5F_6$, as required.

Part (b): Bruckman's analysis of part (b) was similar, yielding the unique solution

$$(a,b,c)=(L_0L_1L_2,L_0L_3,L_1^2+L_2^2)=(2\cdot 1\cdot 3,2\cdot 4,1^2+3^2)=(6,8,10).$$

In this case, the area of the triangle formed by a, b, and c is equal to $\frac{1}{2}\cdot 6\cdot 8=24=2\cdot 1\cdot 3\cdot 4=L_0L_1L_2L_3$, the product of four consecutive Lucas numbers, as required.

Some solvers found a solution but did not prove that it was unique. Thus, they did not technically prove that the area of the triangle must be the product of four consecutive Fibonacci numbers.

Also solved by Charles Ashbacher, Leonard A. G. Dresel; Jane Friedman, Marquis Griffith & Ryan Jackson (jointly); Russell Jay Hendel, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Editorial Note: *Problems B-707 and B-708 were also solved by Igor Ol. Popov.*

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