HYPERSPACES AND FIBONACCI NUMBERS

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Let $(\mathscr{F}([0,1]), \tau_V)$ be the space of all closed non-empty subsets of the closed interval $[0,1]$ equipped with the Vietoris topology. The basic open subsets of τ_V are given by

$$\mathscr{F}_{G_1,\ldots,G_n} = \left\{A \in [0,1];\ A \neq \varnothing \text{ and closed such that } A \subseteq \bigcup_{i=1}^n G_i,\ G_i \cap A \neq \varnothing \forall i\right\}$$

for any collection $G_1, \ldots, G_n$ of nonempty open subsets of $[0,1]$.

Now let $G_1, \ldots, G_m$ $(m \in N)$ be open intervals of 0, 1] satisfying

(1) $$[0,1] = \bigcup_{i=1}^m G_i$$

(2) $$G_i \cap G_{i+1} \neq \varnothing \forall i = 1, \ldots, m-1$$

(3) $$G_i \cap G_j = \varnothing \text{ for } |i-j| \geq 2.$$

The main purpose of this paper is to calculate the number

(4) $$n(G_1, \ldots, G_m) = \max_A \left|\left\{\mathscr{F}_{(G_i)_{i\in I}};\ A \in \mathscr{F}_{(G_i)_{i\in I}},\ I \subseteq \{1, \ldots, m\}\right\}\right|$$

where A ranges over $\mathscr{F}([0,1])$ and where each $\mathscr{F}_{(G_i)_{i\in I}}$ is a basic open set of τ_V. Obviously, $n(G_1) = 1$; we shall investigate the case where $m \geq 2$. This problem was raised by the attempt to find a Hausdorff function h with zero local measure [2, 3], but non-σ-finite Hausdorff measure [1] for $\mathscr{F}([0,1])$. The calculation uses Fibonacci numbers [4].

Lemma 1: There exists $A_0 \in \mathscr{F}([0,1])$ which gives a maximum for (4) such that

(5) $$A_0 \subseteq \bigcup_{i\neq j}(G_i \cap G_j).$$

Proof: Without loss of generality, we may assume

$$|A_0 \cap (G_i \cap G_{i+1})| \leq 1 \text{ for } i = 1, \ldots, m-1$$

and

$$|A_0 \cap G_i \setminus (G_{i-1} \cup G_{i+1})| \leq 1 \text{ for } i = 2, \ldots, m-1$$

as well as

$$|A_0 \cap (G_1 \setminus G_2)| \leq 1 \text{ and } |A_0 \cap (G_m \setminus G_{m-1})| \leq 1.$$

Now let $i_0 \in \{2, \ldots, m-1\}$ (the cases $i_0 = 1$ and $i_0 = m$ can be handled in the same way) and suppose that

$$\{x\} = A_0 \cap G_{i_0} \setminus (G_{i_0-1} \cup G_{i_0+1})$$

for a set A_0 giving the maximum for (4). If $A_0 \in \mathscr{F}_{(G_i)_{i\in I}}$, then $i_0 \in I$. Hence, $A_0 \setminus \{x\}$ would also give the maximum. Deleting all such points, we obtain the desired set. □

Now let $x_i \in G_i \cap G_{i+1}$ for $i = 1, \ldots, m-1$ and $E = \{x_i; i = 1, \ldots, m-1\}$.

Lemma 2: For $m \geq 2$,

$$\left|\left\{\mathcal{F}_{(G_i)_{i\in I}};\ E \in \mathcal{F}_{(G_i)_{i\in I}},\ I \subseteq \{1, \ldots, m\}\right\}\right| = u_{m+2}$$

where u_{m+2} is the $(\mathrm{m}+2)^{\text{th}}$ Fibonacci number, i.e., $u_1 = u_2 = 1$ and $u_{m+1} = u_m + u_{m-1}$ for $m \geq 2$ (see [4]).

Proof: Clearly $E \in \mathcal{F}_{(G_i)_{i\in I}}$ for $I \subseteq \{1, \ldots, m\}$ if and only if

(6) $$i \notin I,\ i \notin \{1, \ldots, m-1\} \text{ implies } i+1 \in I.$$

Let us now consider the hypergraphs

$$\mathcal{A}_m = \{I \subseteq \{1, \ldots,\};\ I \text{ satisfies } (6)\}$$

and

$$\mathcal{B}_m = \{I \subseteq \{1, \ldots, m\};\ m \in I \text{ and } I \text{ satisfies } (6)\}.$$

We see that

(7) $$\mathcal{A}_{m+1} = \{I \cup \{m+1\};\ I \in \mathcal{A}_m\} \cup \mathcal{B}_m.$$

Since $\{I \cup \{m+1\};\ I \in \mathcal{A}_m\} \cap \mathcal{B}_m = \varnothing$, it follows that

(8) $$|\mathcal{A}_{m+1}| = |\mathcal{A}_m| + |\mathcal{B}_m|$$

using the fact that $|\{I \cup \{m+1\};\ I \in \mathcal{A}_m\}| = |\mathcal{A}_m|$. We partition $\mathcal{B}_{m+1}$ as follows:

(9) $$\mathcal{B}_{m+1} = \{I \in \mathcal{B}_{m+1};\ m \in I\} \cup \{I \in \mathcal{B}_{m+1};\ m \notin I\}.$$

It is

(10) $$|\{I \in \mathcal{B}_{m+1};\ m \in I\}| = |\mathcal{B}_m|.$$

If $I \in \mathcal{B}_{m+1}$ and $m \notin I$, then $m-1 \in I$, implying that

(11) $$|\{I \in \mathcal{B}_{m+1};\ m \notin I\}| = |\mathcal{B}_{m+1}|.$$

Because of $\{I \in \mathcal{B}_{m+1};\ m \in I\} \cap \{I \in \mathcal{B}_{m+1};\ m \notin I\} = \varnothing$, we obtain

(12) $$|\mathcal{B}_{m+1}| = |\mathcal{B}_m| + |\mathcal{B}_{m+1}|.$$

We conclude, with $|\mathcal{B}_1| = 1$ and $|\mathcal{B}_2| = 2$, that

(13) $$|\mathcal{A}_{m+1}| = 1 + \sum_{k=1}^{m} |\mathcal{B}_k|$$

for $m \geq 1$ using (8). This gives, together with (12), that

(14) $$|\mathcal{A}_{m+1}| = |\mathcal{B}_{m+2}|.$$

Let $(u_m)_{m\in N}$ be the sequence of Fibonacci numbers with $u_1 = u_2 = 1$ and $u_{m+2} = u_{m+1} + u_m$, then it is easy to see that

(15) $$|\mathcal{A}_m| = u_{m+2}. \ \square$$

Lemma 3:

(16) $$u_{m+k+2} \le u_{m+2}u_{k+2} \text{ for } m \ge 2 \text{ and } k \ge 1.$$

In particular, $u_{k+4} < 3u_{k+2}$ for $k \ge 1$.

Proof: Using the well-known relation $u_{i+j} = u_{i-1}u_j + u_i u_{j+1}$ [4] with $i = m+1,\ j = k+1$ we obtain from $u_{k+1} < u_{k+2}$ that

$$u_m u_{k+1} < u_m u_{k+2}$$
$$u_m u_{k+1} < (u_{m+2} - u_{m+1})u_{k+2}$$
$$u_m u_{k+1} + u_{m+1}u_{k+2} < u_{m+2}u_{k+2}$$
$$u_{m+k+2} < u_{m+2}u_{k+2}. \quad \square$$

Theorem 1: For $m \ge 2$,

$$n(G_1, \dots, G_m) = \begin{cases} 3^{\frac{m}{2}} & \text{if } m \text{ is even} \\ 5(3^{\frac{m-3}{2}}) & \text{if } m \text{ is odd.} \end{cases}$$

Proof: Let $A_0 \subseteq [0, 1]$ be a finite set with property (5) giving maximum. Then

(17) $$A_0 = \textstyle\bigcup_{j=1}^{\ell} A_{0,j} \quad (\ell < m)$$

with $A_{0,j} = \{x_{i_j}, x_{i_j+1}, \dots, x_{i_j+k_j}\}$ such that

(18) $$i_j + k_j + 1 < i_{j+1} \text{ for } 1 \le j \le \mathsf{l} - 1.$$

Let $n_{A_0} = n(G_1, \dots, G_m)$ and $n_{A_{0,j}}$ the number of $\mathcal{F}_{(G_i)_{i\in I}}$ which contain $A_{0,j}$. The condition (18) guarantees that a set $\mathcal{F}_{(G_i)_{i\in I}}$ containing $A_{0,j}$ cannot contain another $A_{0,j'}$ with $j \ne j'$. Now, let

$$\left(\mathcal{F}_{(G_i^{(j)})_{i\in I_j}}\right)_{j=1}^{\ell}$$

be any collection which contains $A_{0,1}, \dots, A_{0,\mathsf{l}}$; then $\mathcal{F}_{(G_i^{(j)})_{i\in I_j}},\ j \in \{1, \dots, \ell\}$ contains A_0. Conversely, we can split the collection $(G_i)_{i\in I}$ if $\mathcal{F}_{(G_i)_{i\in I}}$ contains A_0 into l subcollections giving families of sets $\mathcal{F}_{(G_i^{(j)})_{i\in I_j}}$ for $j \in \{1, \dots, \ell\}$ which contain $A_{0,j}$ by using (18) again. Thus, we conclude that

$$n_{A_0} = \prod_{j=1}^{\ell} n_{A_{0,j}} \quad (\text{where } n_{A_{0,j}} = u_{k_j+4}).$$

We claim now that $|A_{0,j}| =$ with the possible exception of one index j for which $|A_{0,j}| = 2$. We claim, moreover, that there is a gap of one point x_i between $A_{0,j}$ and $A_{0,j+1}$ for $j = 1, \dots, \ell - 1$. First it should be clear that for a set A_0 giving maximum for $n(G_1, \dots, G_m)$ the gap between its components $A_{0,j}$ and $A_{0,j+1}$ is at most one point. If there is a component $A_{0,j}$ with $|A_{0,j}| \ge 3$ then we could delete, say x_{i_j+1} and the set $A_0 \backslash \{x_{i_j+1}\}$ is contained in

$$\frac{n(G_1, \ldots, G_m)}{u_{k_j+4}}(3u_{k_j+2})$$

sets $\mathcal{F}_{(G_i)_{i\in I}}$ which contradicts the assumption that A_0 already gives the maximum by using Lemma 3. Hence $|A_{0,j}| = 1$ or 2 for all j. Assuming that $|A_{0,j}| = |A_{0,j+1}| = 2$, then

$$(A_0 \setminus \{x_{i_j+1}, x_{i_{j+1}}\}) \cup \{x_{i_j+2}\}$$

is contained in more sets $\mathcal{F}_{(G_i)_{i\in I}}$ than A_0 since $A_{0,j}$ and $A_{0,j+1}$ give the factor 5 for the resulting product $n(G_1, \ldots, G_m)$. but three points give the factor 3 and 25 < 27. Furthermore, one can change the order of the $A'_{0,j}$'s to a consecutive one for the sets $A_{0,j}$ with $|A_{0,j}| = 2$. Thus, there exists at most one $A_{0,j}$ with $|A_{0,j}| = 2$. If m is even, then $|A_0| = \frac{m}{2}$ with $\frac{m-2}{2}$ gap points, i.e., $|A_{0,j}| = 1$ for all components of A_0, but the last component contains two points. This gives the announced result for $n(G_1, \ldots, G_m)$. □

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Editor on Leave of Absence

The Editor has been asked to visit Yunnan Normal University in Kunming, China for the Fall semester of 1993. This is an opportunity which the Editor and his wife feel cannot be turned down. They will be in China from August 1, 1993 until around January 10, 1994. The August and November issues of *The Fibonacci Quarterly* will be delivered to the printer early enough so that these two issues can still be published while the Editor is gone. The Editor has also arranged for several individuals to send out articles to be refereed which have been submitted for publication in *The Fibonacci Quarterly* or submitted for presentation at the ***Sixth International Conference on Fibonacci Numbers and Their Applications.*** Things may be a little slower than normal but every attempt will be made to insure that things go as smoothly as possible while the Editor is on leave in China. PLEASE CONTINUE TO USE THE NORMAL ADDRESS FOR SUBMISSION OF PAPERS AND ALL OTHER CORRESPONDENCE.