CARLITZ GENERALIZATIONS OF LUCAS AND LEHMER SEQUENCES

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1. INTRODUCTION

A Lucas fundamental sequence $\{u_n\}_{n=0}^{\infty}$ is a nondegenerate binary recurrence sequence with initial conditions $u_0 = 1$, $u_1 = P$ which satisfies the homogeneous second-order linear recurrence relation

(1.1) $$u_n = Pu_{n-1} - Qu_{n-2}, \quad n \geq 2,$$

where P and Q are integers [12].

If the associated auxiliary equation

(1.2) $$x^2 - Px + Q = 0$$

has roots α, β, then

(1.3) $$u_n = (\alpha^{n+1} - \beta^{n+1}) / (\alpha - \beta).$$

The Fibonacci, Mersenne, and Fermat numbers are all types of Lucas numbers. Their properties were studied extensively by Carmichael [5].

Many authors have generalized aspects of them by various alterations to the characteristic equations. Some of these may be found in Dickinson [6], Feinberg [7], Harris & Styles [8], Horadam [10], Miles [14], Raab [15], Williams [19], and Zeitlin [20]. Atanassov et al. [1] have coupled the recurrence relations in their generalizations.

Lehmer [11] generalized the results of Lucas on the divisibility properties of Lucas numbers to numbers

(1.4) $$\ell_n = \begin{cases} (\alpha^n - \beta^n) / (\alpha - \beta), & \text{for } n \text{ odd} \\ (\alpha^n - \beta^n) / (\alpha^2 - \beta^2), & \text{for } n \text{ even.} \end{cases}$$

It is a generalization of these numbers that we wish to consider in this paper. It is of interest to note in passing that McDaniel has also recently studied analogies between the Lucas and Lehmer sequences [13].

2. DEFINITIONS

Following Carlitz [4], we define

(2.1) $$f_n^{(r)} = (\alpha^{nk+k} - \beta^{nk+k}) / (\alpha^k - \beta^k)$$

and

(2.2) $$g_n^{(r)} = (\alpha^{n+k} - \beta^{n+k}) / (\alpha^k - \beta^k)$$

which are not necessarily integers, where $k = r - 1$, and α and β are the roots of (1.2) as before. For example,

$$f_n^{(2)} = (\alpha^{n+1} - \beta^{n+1}) / (\alpha - \beta) = g_n^{(2)} = u_n$$

so that these numbers are generalizations of the Lucas numbers. They are also generalizations of the Lehmer numbers if we let

$$\ell_n = \begin{cases} f_{n-1}^{(2)}, & \text{for } n \text{ odd,} \\ g_{n-2}^{(3)}, & \text{for } n \text{ even,} \end{cases} \tag{2.3}$$

Carlitz [4] first defined the $f_n^{(r)}$ in another context and proved that

$$f_n^{(r)} = tr(A_{n+1}^r)$$

where

$$A_{n+1} = \left[\binom{t}{n-s}\right] \quad (t, s = 0, 1, \dots n)$$

is a matrix of order $n + 1$.

Examples of $f_n^{(r)}$ and $g_n^{(r)}$ now follow. When $P = -Q = 1$ and $r = 2, 3, 4$, in turn, we have:

n	0	1	2	3	4	5	6	...
$f_n^{(2)}$	1	1	2	3	5	8	13	...
$f_n^{(3)}$	1	3	8	21	55	144	377	...
$f_n^{(4)}$	1	4	17	72	305	1292	5473	...
$g_n^{(2)}$	1	1	2	3	5	8	13	...
$g_n^{(3)}$	1	2	3	5	8	13	21	...
$g_n^{(4)}$	1	$\frac{3}{2}$	$\frac{5}{2}$	4	$\frac{13}{2}$	$\frac{21}{2}$	17	...

It can be seen from this table and the recurrence relations that other properties for these sequences could be developed by treating them as cases of Horadam's $Py_{k-1} + P^2 y_{k-3} = v_k \cdot \{w_n\}$ [10].

3. RECURRENCE RELATIONS

We also need the Lucas primordial sequence $\{v_n\}_{n=0}^{\infty}$ defined by the recurrence relation (1.1) with initial terms $v_0 = 2$ and $v_1 = P$, so that the general term is given by

$$v_n = \alpha^n + \beta^n. \tag{3.1}$$

We can show that

$$f_{n+1}^{(r)} = v_{r-1} f_n^{(r)} - Q^{r-1} f_{n-1}^{(r)} \tag{3.2}$$

and

$$g_{n+1}^{(r)} = v_1 g_n^{(r)} - Q g_{n-1}^{(r)}. \tag{3.3}$$

The latter is the same as (1.1) when $v = 2$ since $v_1 = P$.

Proof of (3.2):

$$\begin{aligned}v_{r-1}f_n^{(r)} - Q^{r-1}f_{n-1}^{(r)} &= \left((\alpha^k+\beta^k)(\alpha^{nk+k}-\beta^{nk+k}) - (\alpha\beta)^k(\alpha^{nk}-\beta^{nk})\right)\Big/(\alpha^k-\beta^k)\\ &= \left(\alpha^{nk+2k}-\beta^{nk+2k}+(\alpha\beta)^k(\alpha^{nk}-\beta^{nk})-(\alpha\beta)^k(\alpha^{nk}-\beta^{nk})\right)\Big/(\alpha^k-\beta^k)\\ &= (\alpha^{nk+2k}-\beta^{nk+2k})/(\alpha^k-\beta^k)\\ &= f_{n+1}^{(r)}, \text{ as required.}\end{aligned}$$

Proof of (3.3):

$$\begin{aligned}v_1 g_n^{(r)} - Qg_{n-1}^{(r)} &= \left((\alpha+\beta)(\alpha^{n+k}-\beta^{n+k}) - (\alpha\beta)(\alpha^{n+k-1}-\beta^{n+k-1})\right)\Big/(\alpha^k-\beta^k)\\ &= \left(\alpha^{n+k+1}-\beta^{n+k+1}+(\alpha\beta)(\alpha^{n+k-1}-\beta^{n+k-1})-(\alpha\beta)(\alpha^{n+k-1}-\beta^{n+k-1})\right)\Big/(\alpha^k-\beta^k)\\ &= (\alpha^{n+k+1}-\beta^{n+k+1})/(\alpha^k-\beta^k)\\ &= g_{n+1}^{(r)}, \text{ as required.}\end{aligned}$$

Thus, the ordinary generating functions will be given (formally) by

$$\sum_{n=0}^{\infty} f_n^{(r)}x^n = 1/(1-v_{r-1}x+Q^{r-1}x^2), \tag{3.4}$$

and

$$\sum_{n=0}^{\infty} g_n^{(r)}x^n = \left(1-Q\left(\frac{u_{r-3}}{u_{r-2}}\right)x\right)\Big/(1-v_1x+Qx^2). \tag{3.5}$$

Proof of (3.4):

$$\begin{aligned}(1-v_{r-1}x+Q^{r-1}x^2)\sum_{n=0}^{\infty} f_n^{(r)}x^n &= f_0^{(r)} + \left(f_1^{(r)} - f_0^{(r)}v_{r-1}\right)x \qquad \text{[by (3.2)]}\\ &= 1+\left(\frac{\alpha^{2k}-\beta^{2k}}{\alpha^k-\beta^k} - (\alpha^k+\beta^k)\right)x \quad \text{[by (2.1)]}\\ &= 1.\end{aligned}$$

Proof of (3.5):

$$\begin{aligned}(1-v_1x+Qx^2)\sum_{n=0}^{\infty} g_n^{(r)}x^n &= g_0^{(r)} + \left(g_1^{(r)} - g_0^{(r)}v_1\right)x\\ &= 1+\left(\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha^k-\beta^k} - (\alpha+\beta)\right)x\\ &= 1-Q\left(\frac{u_{k-2}}{u_{k-1}}\right)x, \text{ as required.}\end{aligned}$$

The $g_n^{(r)}$ are related to the $f_n^{(r)}$ by

$$g_{kn}^{(r)} = f_n^{(r)}, \tag{3.6}$$

and to the Lucas primordial numbers v_n by

(3.7) $$v_n = Pg_{n-1}^{(r)} - Qv_{r-2}g_{n-k-1}^{(r)}$$

and

(3.8) $$v_n = g_{2n-k}^{(r)} / g_{n-k}^{(r)}.$$

Proof of (3.7):

$$\begin{aligned}Pg_n^{(r)} - Qv_{r-2}g_{n-k}^{(r)} &= \left((\alpha+\beta)(\alpha^{n+k}-\beta^{n+k}) - (\alpha\beta)(\alpha^{k-1}+\beta^{k-1})(\alpha^n-\beta^n)\right)\big/(\alpha^k-\beta^k)\\ &= (\alpha^{n+k+1}-\beta^{n+k+1}+\alpha^k\beta^{n+1}-\alpha^{n+1}\beta^k)/(\alpha^k-\beta^k)\\ &= \left(\alpha^{n+1}(\alpha^k-\beta^k)+\beta^{n+1}(\alpha^k-\beta^k)\right)\big/(\alpha^k-\beta^k)\\ &= v_{n+1}, \text{ as required.}\end{aligned}$$

Proof of (3.8):

$$\begin{aligned}g_{n-k}^{(r)}v_n &= (\alpha^n-\beta^n)(\alpha^n+\beta^n)/(\alpha^k-\beta^k)\\ &= (\alpha^{2n}-\beta^{2n})/(\alpha^k-\beta^k)\\ &= g_{2n-k}^{(r)}, \text{ as required.}\end{aligned}$$

4. GENERALIZATIONS OF BARAKAT'S RESULTS

As an analog of Simson's relation, we have

$$(g_n^{(r)})^2 - g_{n-k}^{(r)}g_{n+k}^{(r)} = Q^n.$$

Proof: The numerator of the left-hand side reduces to

$$\begin{aligned}(\alpha\beta)^n\alpha^{2k}+(\alpha\beta)^n\beta^{2k}-2(\alpha\beta)^n(\alpha\beta)^k &= (\alpha\beta)^n(\alpha^k-\beta^k)^2\\ &= Q^n(\alpha^k-\beta^k)^2\end{aligned}$$

which is Q^n times the denominator of the left-hand side.

When $P=-Q$, we are able to relate the $f_n^{(r)}$ to the ordinary Lucas fundamental numbers, u_n, by means of a generalization of a result of Barakat [2] for the ordinary Lucas fundamental numbers. Barakat proved

$$\begin{aligned}u_n &= \sum_{0\le 2m\le n}\binom{n-m}{m}P^{n-2m}(-Q)^m\\ &= \sum_{0\le 2m\le n}\binom{n-m}{m}P^{n-m} \text{ when } P=-Q.\end{aligned}$$

We define

$$x_n = u_n/P = (\alpha^{n+1}-\beta^{n+1})/(\alpha^2-\beta^2),$$

and, for notational convenience, set $y_n = x_{n+1}$. Thus, from Simson's relation, which can be expressed as $u_k^2 - u_{k-1}u_{k+1} = (-P)^k$, we have

(4.2) $$x_ky_{k-1} - x_{k-1}y_k = (-P)^{k-2}.$$

Proof: If we divide the left-hand side of Simson's relation by P^2, we get

$$x_k y_{k-1} - x_{k-1} y_k = (-P)^{k-2}$$

from which the result follows.

A variation of equation (3.7) is then

$$Py_{k-1} + P^2 y_{k-3} = v_k. \tag{4.3}$$

Proof: The numerator of the left-hand side is

$$\begin{aligned}
&(\alpha+\beta)(\alpha^{k+1}-\beta^{k+1}) + (-\alpha\beta)(\alpha+\beta)(\alpha^{k-1}-\beta^{k-1})\\
&= \alpha^{k+2} - \beta^{k+2} - \alpha\beta^{k+1} + \alpha^{k+1}\beta - \alpha^{k+1}\beta + \alpha^2\beta^k - \alpha^k\beta^2 + \alpha\beta^{k+1}\\
&= (\alpha^k + \beta^k)(\alpha^2 - \beta^2)\\
&= v_k \cdot (\alpha^2 - \beta^2) \quad \text{which gives the result (4.3).}
\end{aligned}$$

We are now in a position to assert a property which relates these generalized Fibonacci numbers to ordinary Fibonacci numbers and at the same time gives an iterative formula for the general term. This formula generalizes Barakat [2] and Shannon [16].

$$f_n^{(r)} = \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} u_{k-2}^{m-s} u_{k-1}^{2s} u_k^{n-m-s} P^m.$$

Proof:

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n^{(r)} z^n &= (1 - v_k z + (-P)^k z^2)^{-1} && \text{[from (3.4)]}\\
&= (1 - (P^2 y_{k-3} + P y_{k-1})z + (-P)^k z^2)^{-1} && \text{[from (4.3)]}\\
&= (1 - (P^2 x_{k-2} + P y_{k-1})z + P^3(x_{k-2}y_{k-1} - x_{k-1}y_{k-2})z^2)^{-1} && \text{[from (4.2)]}\\
&= ((1 - \mathrm{P}^2 x_{k-2} z)(1 - P y_{k-1} z) - P^3 x_{k-1} y_{k-2} z^2)^{-1}\\
&= \sum_{s=0}^{\infty} (1 - P^2 x_{k-2} z)^{-s-1} (1 - P y_{k-1} z)^{-s-1} x_{k-1}^s y_{k-2}^s P^{3s} z^{2s}\\
&= \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \binom{m+s}{s} (1 - P y_{k-1} z)^{-s-1} x_{k-1}^s x_{k-2}^m y_{k-2}^s P^{3s+2m} z^{2s+m}\\
&= \sum_{m=0}^{\infty} \sum_{s=0}^{m} \binom{m}{s} (1 - P y_{k-1} z)^{-s-1} x_{k-1}^s x_{k-2}^{m-s} y_{k-2}^s P^{s+2m} z^{m+s}\\
&= \sum_{m=0}^{\infty} \sum_{s=0}^{m} \sum_{t=0}^{\infty} \binom{m}{s}\binom{t+s}{s} x_{k-1}^s x_{k-2}^{m-s} y_{k-1}^t y_{k-2}^s P^{s+2m+t} z^{m+s+t}\\
&= \sum_{n=0}^{\infty} \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} x_{k-1}^s x_{k-2}^{m-s} y_{k-1}^{n-m-s} y_{k-2}^s P^{n+m} z^n.
\end{aligned}$$

So by equating coefficients of z^n we find

$$f_n^{(r)} = \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} \frac{u_{k-1}^{2s} u_{k-2}^{m-s} u_k^{n-m-s} P^{n+m}}{P^{s+m-s+n-m-s+s}}$$

$$= \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} u_{k-2}^{m-s} u_{k-1}^{2s} u_k^{n-m-s} P^m, \text{ as required.}$$

For example, when $r = 2$ (and so $k = 1$), since $f_{-1}^{(2)} = 0$,

$$f_n^{(2)} = \sum_{0 \le 2m \le n} \binom{n-m}{m} P^{n-2m} P^m$$

$$= \sum_{0 \le 2m \le n} \binom{n-m}{m} P^{n-m},$$

which agrees with the result due to Barakat above. Note that Bruckman [3] has provided a neater proof for (4.3) in the case when $P = 1$.

5. CONCLUDING COMMENTS

Other properties can be readily developed to relate the $f_n^{(r)}$ and $g_n^{(r)}$ to other parts of the recurrence relation theory. For instance, we can prove that

$$\left\{ n \atop j \right\} \prod_{i=2}^{j+1} g_{n-2i+4}^{(i)} \tag{5.1}$$

where $\left\{ n \atop j \right\}$ is the analogue of the binomial coefficient used extensively in recurrence relation work (for example, Horadam [9]), and defined by

$$\left\{ n \atop j \right\} = \frac{u_n u_{n-1} \cdots u_{n-j-1}}{u_0 u_1 \cdots u_{j-1}}$$

Proof:

$$\left\{ n \atop j \right\} = \frac{(\alpha^{n+1} - \beta^{n+1})(\alpha^n - \beta^n)\cdots(\alpha^{n-j+2} - \beta^{n-j+2})}{(\alpha - \beta)(\alpha^2 - \beta^2)\cdots(\alpha^j - \beta^j)}$$

$$= g_n^{(2)} g_{n-2}^{(3)} \cdots g_{n-2j+2}^{(j+1)} \qquad \text{[by (2.2)]}$$

$$= \prod_{i=2}^{j+1} g_{n-2i+4}^{(i)}.$$

As another instance, consider

$$f_n^{(r)} = \beta^{kn} \left(\underline{(\alpha / \beta)^k} \right)_{n+1} \tag{5.2}$$

where $\underline{x}_n$ represents the n^{th} reduced Fermatian of index x as mentioned by Whitney [18] and utilized by Shannon [17]. It is defined formally by $\underline{x}_n = 1 + x + x^2 + \cdots + x^{n-1}$.

Proof of (5.2):

$$f_n^{(r)} = \frac{(\alpha^k)^{n+1} - (\beta^k)^{n+1}}{(\alpha^k) - (\beta^k)} = \beta^{kn}\left(1 + \left(\left(\frac{\alpha}{\beta}\right)^k\right) + \cdots + \left(\left(\frac{\alpha}{\beta}\right)^k\right)^n\right) = \beta^{kn}\left(\underline{(\alpha/\beta)}^k\right)_{n+1}.$$

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