A DISJOINT SYSTEM OF LINEAR RECURRING SEQUENCES GENERATED BY $u_{n+2} = u_{n+1} + u_n$ WHICH CONTAINS EVERY NATURAL NUMBER

Joachim Zöllner
Wallaustr. 77, D-6500 Mainz, Germany
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Burke and Bergum [1] called a (finite or infinite) family of n^{th}-order linear recurring sequences a (finite or infinite) *regular covering* if every natural number is contained in at least one of these sequences. If every natural number is contained in exactly one of these sequences, they called the family a (finite or infinite) *disjoint covering*. They gave examples of finite and infinite disjoint coverings generated by linear recurrences of every order n. In the case of the Fibonacci recurrence $u_{n+2} = u_{n+1} + u_n$, they constructed a regular covering which is not disjoint and asked whether a disjoint covering in this case exists as well. The following theorem answers this question.

Theorem: There is an infinite disjoint covering generated by the linear recurrence $u_{n+2} = u_{n+1} + u_n$.

We first state some easy properties of the Fibonacci numbers, $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n = 1, 2, \ldots$. Let $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $\beta = \frac{1}{2}(1-\sqrt{5})$. We have

(1) $$\alpha < 1, \quad -1 < \beta < 0$$

and

(2) $$\alpha|\beta| = 1.$$

For the Fibonacci numbers, the Binet formula

(3) $$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad (n \in \mathbf{N})$$

holds.

For all $i \in \mathbf{N}$, let $u_{i,1}, u_{i,2} \in \mathbf{N}$ and the sequences $(u_{i,n})_{n\in\mathbf{N}}$ be defined by

(4) $$u_{i,n+2} = u_{i,n+1} + u_{i,n}.$$

Then we have

(5) $$u_{i,n} = F_{n-1}u_{i,2} + F_{n-2}u_{i,1}$$

and

(6) $$u_{i,n+1} = \alpha u_{i,n} + \beta^{n-2}(\beta u_{i,2} + u_{i,1})$$

for all $i, n \in \mathbf{N}, n \geq 2$.

Proof of the Theorem: We will construct sequences $(u_{i,n})_{n\in\mathbf{N}}$ of natural numbers for all $i \in \mathbf{N}$ generated by (4).

Start with $(u_{i,n})_{n\in\mathbf{N}} = (F_{n+1})_{n\in\mathbf{N}}$ and assume that $(u_{i,n})_{n\in\mathbf{N}}$ has been constructed for $i = 1, 2, \ldots, k-1$ for some $k \in \mathbf{N}$, $k \geq 2$, and that $u_{i,n} = u_{j,m}$ if and only if $m = n$ and $i = j$ $(i < k, j < k)$.

Now we construct $(u_{k,n})_{n\in\mathbf{N}}$ with the same property. Let $V_i = \{u_{j,n} | n \in \mathbf{N}, j = 1, 2, \ldots, i\}$.

By (1), (3), and (4), we have $\mathbf{N} \setminus V_{k-1} \neq \varnothing$. Thus, we can choose

$$u_{k,1} = \min(\mathbf{N} \setminus V_{k-1}). \tag{7}$$

We will show that there are $u_{k,2} \in \mathbf{N}$ with

$$u_{k,2} > u_{k,1} \tag{8}$$

and

$$u_{k,2} > \max\{u_{i,2} | i = 1, 2, \ldots, k-1\}, \tag{9}$$

such that the sequence $(u_{k,n})_{n\in\mathbf{N}}$ generated by (4) has the following property P:

$$\text{If } i < k, \text{ then } u_{k,n} \neq u_{i,m} \text{ for all } n, m \in \mathbf{N}. \tag{P}$$

Let $M_k = \max\{u_{k,1}, u_{1,2}, u_{2,2}, \ldots, u_{k-1,2}\}$. Then $u_{k,2} > M_k$ is equivalent to (8) and (9).

Let $S_k \in \mathbf{R}$ be sufficiently large. More precisely

$$S_k \geq 4\alpha^{-1} u_{k,1} \ \ (>1) \tag{10}$$

and

$$S_k > 5(k-1)\big((\log 4S_k) / \log \alpha\big)^2 + M_k \tag{11}$$

$$(\text{e.g.: } S_k = \big((5(k-1)/\log^2 \alpha)^2 + 1\big) M_k\big).$$

To prove the existence of $u_{k,2} \in (M_k, S_k] \cap \mathbf{N}$ such that $(u_{k,n})_{n\in\mathbf{N}}$ has property (P), we first count the number of those integers $u_{k,2} \in (M_k, S_k] \cap \mathbf{N}$ such that $(u_{k,n})_{n\in\mathbf{N}}$ does not have property (P). For these $u_{k,2}$, there are $m, n \in \mathbf{N}$ and $i \in \{1, 2, \ldots, k-1\}$ with

$$u_{k,n} = u_{i,m}. \tag{12}$$

From (7), (8), and (9), we get $n \geq 2$ and $m \geq 3$. By (5) we can write (12) as follows:

$$F_{n-1} u_{k,2} + F_{n-2} u_{k,1} = F_{m-1} u_{i,2} + F_{m-2} u_{i,1}.$$

We obtain

$$n < m \tag{13}$$

and by (3) also

$$\frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}} u_{k,2} + \frac{\alpha^{n-2} - \beta^{n-2}}{\sqrt{5}} u_{k,1} = \frac{\alpha^{m-1} - \beta^{m-1}}{\sqrt{5}} u_{i,2} + \frac{\alpha^{m-2} - \beta^{m-2}}{\sqrt{5}} u_{i,1}.$$

Since $u_{k,2} \leq S_k$ and $|\beta| < \alpha/2$, we get

$$\begin{aligned} S_k \geq u_{k,2} &\geq \frac{\alpha^{m-1} - |\beta|^{m-1}}{\alpha^{n-1} + |\beta|^{n-1}} u_{i,2} + \frac{\alpha^{m-2} - |\beta|^{m-2}}{\alpha^{n-1} + |\beta|^{n-1}} u_{i,1} - \frac{\alpha^{n-2} + |\beta|^{n-2}}{\alpha^{n-1} - |\beta|^{n-1}} u_{k,1} \\ &\geq \frac{\frac{1}{2}\alpha^{m-1}}{2\alpha^{n-1}} u_{i,2} + \frac{\frac{1}{2}\alpha^{m-2}}{2\alpha^{n-1}} u_{i,1} - 4\alpha^{-1} u_{k,1}. \end{aligned}$$

Observing (10), this implies

$$8S_k + 4(S_k + 4\alpha^{-1}u_{k,1}) \ge \alpha^{m-n-1}(\alpha u_{i,2} + u_{i,1}) > 4g\alpha^{m-n-1}$$

$$2S_k > \alpha^{m-n-1}$$

$$\frac{\log 2S_k}{\log \alpha} > m - n - 1. \tag{14}$$

We have $u_{k,n+1} \ne u_{i,m+1}$. Otherwise we would get from (12) and (4) that $u_{k,\ell} = u_{i,m-n+\ell}$ for all $\ell \in \mathbf{N}$. In particular, $u_{k,1} = u_{i,m-n+1}$ would contradict (7).

Using this and (6), (12), (1), (13), (8), (9), (2), and $u_{k,2} \le S_k$, we get

$$\begin{aligned}
1 &\le |u_{k,n+1} - u_{i,m+1}| \\
&= |\alpha u_{k,n} + \beta^{n-2}(\beta u_{k,2} + u_{k,1}) - \alpha u_{i,m} - \beta^{m-2}(\beta u_{i,2} + u_{i,1})| \\
&\le |\beta|^{n-2}|\beta u_{k,2} + u_{k,1}| + |\beta|^{m-2}|\beta u_{i,2} + u_{i,1}| \\
&\le |\beta|^{n-2}(|\beta u_{k,2}| + |u_{k,1}| + |\beta u_{i,2}| + |u_{i,1}|) \\
&\le |\beta|^{n-2} 4u_{k,2} \le \alpha^{-(n-2)} 4S_k
\end{aligned}$$

$$\alpha^{n-2} \le 4S_k$$

$$n \le \frac{\log 4S_k}{\log \alpha} + 2.$$

Combining this with (14), we obtain

$$m < \frac{\log 2S_k}{\log \alpha} + n + 1 \le \frac{\log 2S_k}{\log \alpha} + \frac{\log 4S_k}{\log \alpha} + 3 \le 3\frac{\log 4S_k}{\log \alpha}. \tag{15}$$

Now we will give an upper bound for the number of triples (n, m, i) such that $u_{k,n} = u_{i,m}$, $1 \le i \le k-1$. In this case (15) holds. First, fix *i* and *m*.

Since $2 \le n < m$, there are at most $m-2$ possible values for *n*. Since

$$3 \le m < (3\log 4S_k)/\log \alpha, \qquad \text{for fixed } i,$$

there are at most

$$\frac{1}{2}\left(\frac{3\log 4S_k}{\log \alpha} - 1\right)\left(\frac{3\log 4S_k}{\log \alpha} - 2\right) \le 5\left(\frac{\log 4S_k}{\log \alpha}\right)^2$$

possible pairs (m, n).

Finally, since $1 \le i \le k-1$, there are at most

$$5(k-1)\left(\frac{\log 4S_k}{\log \alpha}\right)^2$$

possible triples (n, m, i). To each triple such that $u_{k,n} = u_{i,m}$, $1 \le i \le k-1$ belongs exactly one $u_{k,2} \in (M_k, S_k] \cap \mathbf{N}$, because for two different values of $u_{k,2}$ and the fixed value of $u_{k,1}$, the recurrence (4) would give two different values of $u_{k,n}$, both of which cannot be equal to $u_{i,n}$. Consequently, there are at most

$$5(k-1)\left(\frac{\log 4S_k}{\log\alpha}\right)^2$$

values of $u_{k,2} \in (M_k, S_k] \cap \mathbf{N}$ such that $u_{k,n} = u_{i,m}$ for some $n, m, 1 \le i \le k-1$. Therefore, the number of values $u_{k,2} \in (M_k, S_k] \cap \mathbf{N}$ such that $u_{k,n} \ne u_{i,m}$ for all $n, m, 1 \le i \le k-1$ is at least

$$S_k - M_k - 5(k-1)\left(\frac{\log 4S_k}{\log\alpha}\right)^2,$$

which is positive by (11), and hence the choice of such an $u_{k,2}$ is possible.

This induction on k shows that there are infinitely many sequences $(u_{k,n})_{n\in\mathbf{N}}$. Every natural number occurs in one of these sequences by (7). It occurs exactly once by property (P) which holds for these sequences.

REFERENCE

1. J. R. Burke & G. E. Bergum. "Covering the Integers with Linear Recurrences." In *Applications of Fibonacci Numbers*, vol 2, pp. 143-47. Ed. A. N. Philippou et al. Dordrecht: Kluwer Academic Publishers, 1988.

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GENERALIZED PASCAL TRIANGLES AND PYRAMIDS
THEIR FRACTALS, GRAPHS, AND APPLICATIONS

by Dr. Boris A. Bondarenko
Associate member of the Academy of Sciences of the Republic of Uzbekistan, Tashkent

Translated by Professor Richard C. Bollinger
Penn State at Erie, The Behrend College