SYMMETRIC FIBONACCI WORDS*

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In [1] the author studied Fibonacci words; the study was motivated by the consideration of Fibonacci strings and Fibonacci word patterns by Knuth [5] and Turner [6, 7], respectively. It was shown in [1] that all the n^{th} Fibonacci words can be obtained from any particular n^{th} Fibonacci word, for example w_n^0, by shifting in a cyclic way the letters in it. Also it was shown that each of the Fibonacci words w_n^0 $(n \geq 3)$ has a representation as a product of two symmetric words. In this paper, we show that every Fibonacci word has such a representation and that this representation is unique (Theorem 3). Furthermore, we prove that, for each positive integer n that is not a multiple of 3, there is precisely one symmetric Fibonacci word of length F_n, where F_n denotes the n^{th} Fibonacci number, while there are no symmetric Fibonacci words of length F_n if n is a multiple of 3 (Theorem 7).

Let X be an alphabet and let X^* be a free monoid of words over X with identity 1. Denote by $\ell(w)$ the length of a word w. Define the reverse R and the shift T on $X^*/\{1\}$ by

$$R(a_1a_2\ldots a_n) = a_na_{n-1}\ldots a_1,$$
$$T(a_1a_2\ldots a_n) = a_2\ldots a_na_1,$$

where $a_i \in X, 1 \leq i \leq n$.

A word $w \in X^*$ is said to be *symmetric* if $w = 1$ or $R(w) = w$. Let $\mathcal{S}$ denote the set of all symmetric words over X and $\mathcal{S}^2 = \{uv : u, v \in \mathcal{S}\} \setminus \{1\}$. The representations uv and vu where $u, v \in \mathcal{S}$, are considered to be the same if $v = 1$.

Fibonacci words are defined recursively as follows. Fix two distinct letters a and b and put

$$w_1 = a,$$
$$w_2 = b,$$
$$w_3^0 = ba, w_3^1 = ab,$$
$$w_4^{00} = bab, w_4^{01} = bba, w_4^{10} = abb, w_4^{11} = bab.$$

In general, suppose that $n \geq 5$, $r_1, r_2, \ldots, r_n$ is a finite binary sequence and that the words

$$w_{n-2}^{r_1r_2\cdots r_{n-4}}, \quad w_{n-1}^{r_1r_2\cdots r_{n-3}}$$

have been defined. Then set

$$w_n^{r_1r_2\cdots r_{n-2}} = \begin{cases} w_{n-1}^{r_1r_2\cdots r_{n-3}} w_{n-2}^{r_1r_2\cdots r_{n-4}} & \text{if } r_{n-2} = 0, \\ w_{n-2}^{r_1r_2\cdots r_{n-4}} w_{n-1}^{r_1r_2\cdots r_{n-3}} & \text{if } r_{n-2} = 1. \end{cases}$$

For simplicity, we write w_n^0 if $n > 3$ and $r_1 = r_2 = \cdots r_{n-2} = 0$. Each $w_n^{r_1r_2\cdots r_{n-2}}$ is called an n^{th} *Fibonacci word* derived from the initial letters a and b and is known to have length F_n.

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Among all the Fibonacci words, some of them are symmetric but some of them are not. For example, the Fibonacci words *bab*, *babab*, *bababbabbabab* are symmetric while the Fibonacci words *abb*, *bba*, *ababb* are not. Nevertheless, it turns out that each Fibonacci word is a unique product of two symmetric words. To prove this unique representation theorem (Theorem 3 below), we need some known results about Fibonacci words (see [1]) and products of two symmetric words (see [2]). The proof of Lemma 1 can be found in [1].

Lemma 1 (Theorems 4 and 7 and Corollary 12(iv) of [1]):

(a) Each w_n^0 $(n \geq 1)$ is a product of two symmetric words, that is $w_n^0 \in \mathcal{S}^2$.

(b) There are exactly F_n distinct Fibonacci words of length F_n, namely, $T^j(w_n^0)$, $0 \leq j \leq F_n - 1$.

In Theorem 2.4 of [2] it was proved that a word has more than one representation as a product of two symmetric words if and only if it is a power of another word which is itself a product of two symmetric words. The following lemma contains Theorem 2.1 of [2] and only part of the result just mentioned because we do not need to use the full power of it to prove the unique representation theorem. For completeness, we include a proof.

Lemma 2 (Theorems 2.1 and 2.2 of [2]):

(a) $\mathcal{S}^2$ is invariant under T, that is, $T(\mathcal{S}^2) \subset \mathcal{S}^2$.

(b) If a word has more than one representation as a product of two symmetric words, then it is a power of another word. More precisely, if p, r, m are positive integers such that $r < p \leq m$ and if, in the word $w = a_1a_2\ldots a_m$, the subwords

$$\begin{array}{l} a_1a_2\ldots a_p,\ a_{p+1}\ldots a_m \\ a_1a_2\ldots a_r,\ a_{r+1}\ldots a_m \end{array} \tag{1}$$

are symmetric words, then $w = (a_1a_2\ldots a_d)^{m/d}$ where $d = (p-r, m)$.

Proof: ***(a)*** If $w = a_1a_2\ldots a_m$ is a symmetric word, then

$$Tw = \begin{cases} a_1 & m = 1, \\ a_2a_1 & m = 2, \\ (a_2\ldots a_{m-1})(a_ma_1) & m > 2. \end{cases}$$

If $w = (a_1a_2\ldots a_p)(a_{p+1}\ldots a_m)$ where p is a positive integer less than m, and the words $a_1a_2\ldots a_p$ and $a_{p+1}\ldots a_m$ are symmetric, then

$$Tw = \begin{cases} (a_2\ldots a_m)a_1 & p = 1, \\ a_2a_3\ldots a_ma_1 & p = 2, \\ (a_2\ldots a_{p-1})(a_pa_{p+1}\ldots a_ma_1) & p > 2. \end{cases}$$

Therefore, ***(a)*** follows.

(b) First, note that since the subwords in (1) are symmetric, we have

$$a_k = a_{p+1-k} = a_{r+1-k} \quad (k = 1, 2, \ldots, m)$$

with indices modulo m. Hence

$$a_k = a_{p-r+k} \quad (k = 1, 2, \ldots, m) \tag{2}$$

with indices modulo m. Now choose positive integers i and j such that $i(p-r)-jm=d$. Then, according to (2), we have

$$a_k = a_{i(p-r)+k} = a_{jm+d+k} = a_{d+k} \quad (k = 1, 2, \ldots, m)$$

with indices modulo m. This proves ***(b)***.

Theorem 3 (Unique representation theorem): Every Fibonacci word has a unique representation as a product of two symmetric words.

Proof: Lemma 1 and Lemma 2(a) imply that every Fibonacci word belongs to $\mathcal{S}^2$. Suppose that some Fibonacci word w has more than one representation as a product of two symmetric words. Then, Lemma 2(b) implies that $w = u^c$ for some word u and $c \geq 2$. But then $T^{\ell(u)}w = w$. Since $1 \leq \ell(u) < \ell(w)$, this contradicts Lemma 1(b). This proves the theorem.

Now we determine all the symmetric Fibonacci words. Let

$$s_n = \begin{cases} 1 & \text{if } n \text{ is a multiple of } 3, \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$t_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Let $p_1 = a, p_2 = b, p_n = w_n^{s_1 s_2 \cdots s_{n-2}}$, for $n \geq 3$, and let $q_n = w_n^{t_1 t_2 \cdots t_{n-2}}$, for $n \geq 3$. For odd n, let $s = F_{n-2}$ and $t = F_{n-1}$; for even n, let $s = F_{n-1}$ and $t = F_{n-2}$.

For $n > 2$, let us list the F_n Fibonacci words of length F_n in the following order (Corollary 12(iv) of [1]):

$$T^0 q_n,\ T^s q_n, \ldots, T^{(F_n - 1)s} q_n. \tag{3}$$

If n is a multiple of 3, then the number of terms in (3) is even, it will be shown in Theorem 7 that there are no symmetric words in the list; however, if n is not a multiple of 3, the number of terms in (3) is odd and, again, it will be shown in Theorem 7 that only the middle term of (3) is a symmetric Fibonacci word.

Lemma 4: If $n > 2$ is not a multiple of 3, then $p_n = T^{js} q_n$ where $j = (F_n - 1)/2$. In other words, p_n is the middle term of the sequence (3).

Proof: As was proved in section 5 of [1], $p_n = T^{js} q_n$ where

$$j \equiv \begin{cases} mF_{n-1} & \text{if } n \text{ is odd,} \\ mF_{n-1} - 1 & \text{if } n \text{ is even,} \end{cases} \pmod{F_n} \tag{4}$$

where $m = 1 + \sum_{i=1}^{n-2} F_{i+1} s_i$. It follows from the identity $F_1 + F_4 + F_7 + \cdots F_{3k-2} = F_{3k}/2$ $(k \geq 1)$ that

$$m = \begin{cases} \frac{1}{2}F_{n-1} & \text{if } n \equiv 1 \pmod 3, \\ \frac{1}{2}F_{n+1} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Thus, if $n \equiv 1 \pmod 3$, then

$$j \equiv (F_{n-2}F_n - 1)/2 \equiv F_n(F_{n-2} - 1)/2 + (F_n - 1)/2 \equiv (F_n - 1)/2 \pmod{F_n};$$

if $n \equiv 2 \pmod 3$, then

$$j \equiv (F_n^2 - 1)/2 \equiv F_n(F_n - 1)/2 + (F_n - 1)/2 \equiv (F_n - 1)/2 \pmod{F_n}.$$

This proves the lemma.

Lemma 5 (Corollary 12(i) of [1]): Let n be a positive integer greater than 2 and $1 \le j \le F_n - 1$. Then the k^{th} letter in $T^{js}q_n$ is an "a" if and only if $k \equiv (j+r)t \pmod{F_n}$ for some $1 \le r \le F_{n-2}$.

Lemma 6: If n is a positive integer greater than 2, then $R(T^{js}q_n) = T^{(F_n-1-j)s}q_n$, for all $0 \le j \le F_n - 1$.

Proof: Let $0 \le j \le F_n - 1$. Suppose that the k^{th} letter in $T^{js}q_n$ is an "a". Then, by Lemma 5, $k \equiv (j+r)t \pmod{F_n}$ for some $1 \le r \le F_{n-2}$. Therefore, $1 \le F_{n-2} + 1 - r \le F_{n-2}$ and

$$\begin{aligned}((F_n - 1 - j) + (F_{n-2} + 1 - r))t &\equiv F_{n-2}t - (j+r)t \\ &\equiv F_{n-2}t - k \equiv F_n + 1 - k \pmod{F_n}.\end{aligned}$$

This proves that $(F_n + 1 - k)^{\text{th}}$ letter in $T^{(F_n-1-j)s}q_n$ is also an "a", again by Lemma 5. Consequently, the result holds.

The above lemma can also be proved by observing that $w_n^{r_1r_2\cdots r_{n-2}} = T^{js}q_n$ where j satisfies (4) with $m = 1 + \sum_{i=1}^{n-2} F_{i+1}r_i$ (section 5 of [1]) and that $R(w_n^{r_1r_2\cdots r_{n-2}}) = w_n^{v_1v_2\cdots v_{n-2}}$, where $v_i = 1 - r_i$, $1 \le i \le n-2$ (Theorem 3(i) of [1]).

Theorem 7: Let n be a positive integer greater than 2.

(a) If n is not a multiple of 3, then p_n is the only symmetric Fibonacci word of length F_n.

(b) If n is a multiple of 3, then no Fibonacci word of length F_n is symmetric.

Proof: Let $0 \le j \le F_n - 1$. Since $F_n - 1 - j = j \Leftrightarrow j = \frac{1}{2}(F_n - 1)$, we see from Lemma 6 that

$$R(T^{js}q_n) = T^{js}q_n \Leftrightarrow j = \tfrac{1}{2}(F_n - 1). \tag{5}$$

(a) If n is not a multiple of 3, then F_n is odd; thus, among the Fibonacci words in (3), $p_n = T^{\frac{1}{2}(F_n-1)s}q_n$ is the only symmetric one, according to (5) and Lemma 4.

(b) If n is a multiple of 3, then, clearly, (5) implies that $T^{js}q_n$ is not symmetric for all $0 \le j \le F_n - 1$.

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