A SUMMATION RULE USING STIRLING NUMBERS OF THE SECOND KIND

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(Submitted September 1991)

1. A SUMMATION RULE

Recall that Stirling numbers of the second kind may be expressed as follows (cf., e.g., [1], [2]):

$$S(m, j) = \frac{1}{j!}\Delta^j 0^m = \frac{1}{j!}\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}i^m,$$

where $\Delta^j 0^m$ is the j^{th} difference of x^m at $x = 0$ so that $S(m, j) = 0$ for $j > m$, $S(m, 0) = 0$ for $m \geq 1$ and $S(0, 0) = 1$.

Summation Rule: Let $F(n, k)$ be a bivariate function defined for integers $n, k \geq 0$. If there can be found a summation formula or a combinatorial identity such as

$$\sum_{k=j}^{n} F(n, k)\binom{k}{j} = \phi(n, j) \quad (j \geq 0), \tag{1}$$

then for every given $m \geq 0$ we have a summation formula or a combinatorial identity such as

$$\sum_{k=0}^{n} F(n, k)k^m = \sum_{j=0}^{m} \phi(n, j)j!S(m, j) \tag{2}$$

which may be called a companion formula of (1).

Generally, (2) would be practically useful when n is much bigger than m.

Proof: It is known that Stirling numbers of the second kind satisfy the following basic relation [which is often taken as a definition of $S(n, k)$]:

$$x^m = \sum_{j=0}^{m} S(m, j)(x)_j, \tag{3}$$

where $(x)_j : x(x-1)\dots(x-j+1)$ $(j \geq 1)$ is the falling factorial with $(x)_0 := 1$. Now, substituting (3) into the left-hand side of (2), changing the order of summation, and using (1), we easily obtain

$$\sum_{k=0}^{n} F(n, k)k^m = \sum_{j=0}^{m} S(m, j)\sum_{k=0}^{n} F(n, k)(k)_j = \sum_{j=0}^{m} j!S(m, j)\phi(n, j).$$

Notice that the special case for $m = 0$ is also true. Hence, (2) holds for every $m \geq 0$. □

Remark Sometimes in applications of the rule function $F(n, k)$ may involve some independent parameters. Moreover, for the particular case in which $F(n, k) > 0$, so that $\phi(n, 0) > 0$, the left-hand side of (2) divided by $\phi(n, 0)$ may be considered as the m^{th} moment (about the origin) of a discrete random variable X that may take possible values 0, 1, 2, ..., n. This means that (2) may sometimes be used for computing moments whenever $F(n, k)/\phi(n, 0)$ just stands for probabilities ($0 \le k \le n$), and the factorial moments $\phi(n, j)/\phi(n, 0)$ are easily found via (1) (cf. David and Barton [3]).

2. VARIOUS EXAMPLES

For the simplest case $F(n, k) \equiv 1$, we have

$$\phi(n, j) \equiv \sum_{k=j}^{n} \binom{k}{j} = \binom{n+1}{j+1}.$$

This leads to the familiar formula

$$\sum_{k=1}^{n} k^m = \sum_{j=1}^{m} \binom{n+1}{j+1} j!\,S(m, j). \tag{4}$$

Actually there are many known identities of type (1) in which $F(n, k)$ may consist of a binomial coefficient or a product of binomial coefficients. See, e.g., Egorychev [4], Gould [5], and Riordan [8]. Consequently, we may find various special summation formulas via (2). We now list a dozen formulas, as follows:

$$\sum_{k=1}^{n} k^m \binom{n}{k} p^k q^{n-k} = \sum_{j=1}^{m} \binom{n}{j} p^j j!\,S(m, j), \tag{5}$$

where $p+q=1$ and $p>0$.

$$\sum_{k=0}^{[n/2]} \binom{n}{2k} k^m = \sum_{j=0}^{m} 2^{n-2j-1} \binom{n-j}{j} \frac{n}{n-j} j!\,S(m, j), \tag{6}$$

$$\sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} k^m = \sum_{j=0}^{m} 2^{n-2j} \binom{n-j}{j} j!\,S(m, j), \tag{7}$$

$$\sum_{k=0}^{n} \binom{n-k}{s} k^m = \sum_{j=0}^{m} \binom{n+1}{s+j+1} j!\,S(m, j), \tag{8}$$

$$\sum_{k=0}^{n} \binom{s+k}{s} k^m = \sum_{j=0}^{m} \binom{n+1}{j} \binom{n+1+s}{s} \frac{n+1-j}{s+1+j} j!\,S(m, j), \tag{9}$$

$$\sum_{k=0}^{n} (-4)^k \binom{n+k}{2k} k^m = \sum_{j=0}^{m} (-1)^n 2^{2j} \binom{n+j}{2j} \frac{2n+1}{2j+1} j!\,S(m, j), \tag{10}$$

$$\sum_{k=0}^{n} (-4)^k \binom{n+k}{2k} \frac{n}{n+k} k^m = \sum_{j=0}^{m} (-1)^n 2^{2j} \binom{n+j}{2j} \frac{n}{n+j} j!\,S(m, j), \tag{11}$$

$$\sum_{k=0}^{[n/2]}(-1)^k\binom{n-k}{k}2^{n-2k}k^m=\sum_{j=0}^{m}(-1)^j\binom{n+1}{2j+1}j!S(m,j), \tag{12}$$

$$\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{n-k}k^m=\sum_{j=0}^{m}\binom{\alpha}{j}\binom{\alpha+\beta-j}{n-j}j!S(m,j), \tag{13}$$

where α and β are real parameters.

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}k^m=\sum_{j=0}^{m}(-1)^j\binom{n}{j}^2 j!S(m,j), \tag{14}$$

$$\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{2k}{k}2^{n-2k}k^m=\sum_{j=0}^{m}\binom{2n-2j}{n}\binom{n}{j}j!S(m,j), \tag{15}$$

$$\sum_{k=1}^{n}k^m H_k=\sum_{j=1}^{m}\binom{n+1}{j+1}\left(H_{n+1}-\frac{1}{j+1}\right)j!S(m,j), \tag{16}$$

where $H_k:=1+\frac{1}{2}+\cdots+\frac{1}{k}$, $(k\geq 1)$, are harmonic numbers.

Though most of the above formulas [except (5)] appear unfamiliar, or are difficult to find in the literature, they are actually companion formulas of some known identities. In fact, (5) is known as the m^{th} moment of the binomial distribution of a discrete random variable. Formulas (6) and (7) represent companion formulas of the pair of Moriarty identities (cf. [4, (2.73) and (2.74)]; [5, (3.120) and (3.121)]). Also, (9) and (12) are just companion formulas of the following identities:

$$\sum_{k=j}^{n}\binom{k+s}{s}\binom{k}{j}=\binom{n+1}{j}\binom{n+1+s}{s}\frac{n+1-j}{s+1+j}$$

and

$$\sum_{k=j}^{[n/2]}(-1)^k\binom{n-k}{k}\binom{k}{j}2^{n-2k}=(-1)^j\binom{n+1}{2j+1}$$

due to Knuth and Marcia Ascher, respectively (cf. [5, (3.155) and (3.179)]). Moreover, (16) may be inferred from the known relation (cf., e.g., [1, pp. 98-99]).

$$\sum_{k=j}^{n}\binom{k}{j}H_k=\binom{n+1}{j+1}\left(H_{n+1}-\frac{1}{j+1}\right). \tag{17}$$

The verification of the rest of the formulas is left to the interested reader.

Evidently, both (8) and (9) imply (4) with $s=0$, and (13) yields the Vandermonde convolution identity when $m=0$. Moreover, it is easily found that (16) leads to an asymptotic relation, for $n\to\infty$, of the following,

$$\sum_{k=1}^{n}k^m H_k\sim\frac{n^{m+1}}{m+1}\left(\log n+\gamma-\frac{1}{m+1}\right),$$

where $\gamma:=\lim_n(H_n-\log n)=0.5772\ldots$ is Euler's constant.

3. AN EXTENSION OF THE SUMMATION RULE

In what follows, we will adopt the notations:

$$(x|h)_n := x(x-h)(x-2h)\cdots(x-nh+h),\ (x|h)_0 = 1,$$

$$\binom{x}{n}_h := (x|h)_n / n!,\ \binom{x}{n}_1 = \binom{x}{n} = (x)_n / n!,\ \binom{x}{n}_0 = x^n / n!.$$

Here $\binom{x}{n}_h$ is known as the generalized binomial coefficient (cf. Jordan [7, ch. 2, §22). Now, suppose that α and β are two distinct real numbers. Consider the following pair of expressions for polynomials $(x|\alpha)_n$ and $(x|\beta)_n$:

$$(x|\alpha)_n = \sum_{k=0}^{n} S_\alpha(n, k|\beta)(x|\beta)_k, \tag{18}$$

$$(x|\beta)_n = \sum_{k=0}^{n} S_\beta(n, k|\alpha)(x|\alpha)_k. \tag{19}$$

The coefficients $S_\alpha(n, k|\beta)$ and $S_\beta(n, k|\alpha)$ involved in (18) and (19) are uniquely determined, and they may be called a pair of symmetrically generalized Stirling numbers associated with the number pair (α, β). Consequently, the ordinary Stirling numbers of the first and second kinds are associated with the number pair (1, 0), and are usually denoted by the following:

$$S_1(n, k) \equiv s(n, k) := S_1(n, k|0),\ S_2(n, k) \equiv S(n, k) := S_0(n, k|1).$$

Certainly, all the well-known properties enjoyed by the ordinary Stirling numbers, e.g., recurrence relations, orthogonality relations, and inversion formulas, etc., can be readily extended to these generalized Stirling numbers. For example, a simple recurrence relation may be deduced from (19), namely

$$S_\beta(n, k|\alpha) = S_\beta(n-1, k-1|\alpha) + (k\alpha - n\beta + \beta)S_\beta(n-1, k|\alpha),\ (k \ge 1). \tag{20}$$

Recall that there is a general form of Newton's expansion for a polynomial $f(x)$ of degree n, viz.,

$$f(x) = \sum_{k=0}^{n} \frac{(x|\alpha)_k}{k!\,\alpha^k} \Delta_\alpha^k f(0), \tag{21}$$

where $\Delta_\alpha^k f(0)$ is the k^{th} difference (with increment α) of $f(x)$ at $x = 0$. Thus, comparing (21) with (19) and (18), we find (with $\alpha\beta \ne 0$),

$$S_\alpha(n, k|\alpha) = \frac{1}{k!\,\alpha^k} \Delta_\alpha^k (x|\beta)_n \Big|_{x=0}, \tag{22}$$

$$S_\alpha(n, k|\beta) = \frac{1}{k!\,\beta^k} \Delta_\beta^k (x|\alpha)_n \Big|_{x=0}. \tag{23}$$

Here, it is easily observed that $S_\beta(n, k|\alpha) = 0$ for $k > n$, and $S_\beta(0, 0|\alpha) = S_\beta(n, n|\alpha) = 1$. Moreover, notice that for $\beta = 0$ (23) should be replaced by

$$S_\alpha(n, k|0) = \frac{1}{k!}\lim_{\beta\to 0}\frac{1}{\beta^k}\Delta_\beta^k(x|\alpha)_n\Big|_{x=0} = \frac{1}{k!}\left(\frac{d}{dx}\right)^k(x|\alpha)_n\Big|_{x=0}.$$

Extended Summation Rule: Let $F(n, k)$ be defined for integers $n, k \geq 0$. If there can be found a summation formula such as

$$\sum_{k=0}^{n} F(n, k)\binom{k}{j}_\alpha = G(n, j),\ (j \geq 0), \tag{24}$$

then for every $m \geq 0$ we have a summation formula of the form

$$\sum_{k=0}^{n} F(n, k)\binom{k}{m}_\beta = \sum_{j=0}^{m} G(n, j)\frac{j!}{m!}S_\beta(m, j|\alpha). \tag{25}$$

Also, suppose that the following series is convergent to $g(j)$ for every $j \geq 0$:

$$\sum_{k=0}^{\infty} f(k)\binom{k}{j}_\alpha = g(j). \tag{26}$$

Then we have a summation formula, as follows:

$$\sum_{k=0}^{\infty} f(k)\binom{k}{m}_\beta = \sum_{j=0}^{m} g(j)\frac{j!}{m!}S_\beta(m, j|\alpha). \tag{27}$$

Proof: Notice that (19) implies

$$\binom{x}{m}_\beta = \frac{1}{m!}\sum_{j=0}^{m} j!\,S_\beta(m, j|\alpha)\binom{x}{j}_\alpha. \tag{28}$$

Thus, both of the implications (24) $\Rightarrow$ (25) and (26) $\Rightarrow$ (27) can be verified in a manner similar to that used to prove (1) $\Rightarrow$ (2). In fact, the verification of (27) can be accomplished by substituting (28) into the left-hand side of (27) and by using (26), in which the change of order of summation is justified by the convergence of the series (26). Moreover, it is evident that

$$S_\alpha(n, k|\alpha) = \begin{cases} 1 & \text{for } k = n, \\ 0 & \text{for } k < n, \end{cases}$$

so that (25) and (27) will transform back to (24) and (26), respectively, when $\beta = \alpha$. Hence, (27) holds for every real number β. □

Examples: For the case $\alpha = 1$, we may write

$$S_\beta(m, j|1) = \frac{1}{j!}\Delta^j(x|\beta)_m\Big|_{x=0}. \tag{29}$$

In particular, we have

$$S_0(m, j|1) = S(m, j),\ S_{-1}(m, j|1) = \frac{m!}{j!}\binom{m-1}{j-1},$$

where $S_{-1}(m, j|1)(-1)^m$ is known as Lah's number.

Making use of the rule (24) $\Rightarrow$ (25) (with $\alpha = 1$), it is readily seen that each of the formulas from (5) through (16) may be generalized to the form in which k^m is replaced by $\binom{k}{m}_\beta$ and $S(m, j)$ by the following: $S_\beta(m, j|1)/m!$. Thus, for instance, (13) and (16) may be replaced, respectively, by:

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}\binom{k}{m}_\beta = \sum_{j=0}^{m}\binom{x+y-j}{n-j}\frac{(x)_j}{m!}S_\beta(m, j|1), \tag{30}$$

$$\sum_{k=1}^{n}\binom{k}{m}_\beta H_k = \sum_{j=1}^{m}\binom{n+1}{j+1}\left(H_{n+1} - \frac{1}{j+1}\right)\frac{j!}{m!}S_\beta(m, j|1). \tag{31}$$

In particular, for $\beta = 0, 1, -1$, we have $\binom{k}{m}_0 = k^m/m!$, $\binom{k}{m}_1 = \binom{k}{m}$, and $\binom{k}{m}_{-1} = \binom{k+m-1}{m}$, so that either (30) or (31) may yield at least three special identities of some interest. Indeed, (31) implies (16), (17), and the identity

$$\sum_{k=1}^{n}\binom{k+m-1}{m}H_k = \sum_{j=1}^{m}\binom{n+1}{j+1}\binom{m-1}{j-1}\left(H_{n+1} - \frac{1}{j+1}\right).$$

Moreover, as a simple consequence of (30), one may take $x = y = n$ and $\beta = 0$ to get

$$\sum_{k=0}^{n}\binom{n}{k}^2 k^m = \sum_{j=0}^{m}\binom{2n-j}{n}(n)_j S(m, j).$$

This is an example mentioned in Comtet [2, ch. 5, p. 225].

To indicate an application of the rule (26) $\Rightarrow$ (27), let us consider the simple example with $f(k) = q^k$:

$$\sum_{k=0}^{\infty}\binom{k}{j}q^k = q^j(1-q)^{-j-1}, \quad (|q| < 1).$$

Consequently, we obtain

$$\sum_{k=0}^{\infty}\binom{k}{m}_\beta q^k = \sum_{j=0}^{m}\frac{q^j \cdot j!}{(1-q)^{j+1}m!}S_\beta(m, j|1). \tag{32}$$

This may be used to evaluate an infinite series involving both generalized binomial coefficients and Fibonacci numbers. Denote $a = \frac{1}{2}(1+\sqrt{5}), b = \frac{1}{2}(1-\sqrt{5})$, and let $\rho > a$. Then the following series,

$$S = \sum_{k=0}^{\infty}\binom{k}{m}_\beta \rho^{-k}F_k,$$

is obviously convergent for every $m \geq 0$, where $f_k = (a^{k+1} - b^{k+1})/\sqrt{5}$. Certainly one may compute the series by means of (32) as follows:

$$S = \frac{\rho}{\sqrt{5}}\sum_{k=0}^{\infty}\binom{k}{m}_\beta\left[(a/\rho)^{k+1} - (b/\rho)^{k+1}\right] = \frac{\rho}{\sqrt{5}}\sum_{j=0}^{m}\left[\left(\frac{a}{\rho-a}\right)^{j+1} - \left(\frac{b}{\rho-b}\right)^{j+1}\right]\frac{j!}{m!}S_\beta(m, j|1).$$

In particular, we have

$$\sum_{k=0}^{\infty} k^m \rho^{-k} F_k = \frac{\rho}{\sqrt{5}} \sum_{j=0}^{m} \left[\left(\frac{a}{\rho - a} \right)^{j+1} - \left(\frac{b}{\rho - b} \right)^{j+1} \right] j!\, S(m, j).$$

Finally, it may be worthy of mention that, for the case $\alpha = 1$, relation (26), apart from the factor $(-1)^j$ just stands for the δ^*-transformation of the given sequence $\{f(k)\}$, which is connected with quasi-Hausdorff transformations (cf. Hardy [6, §11.19]). Moreover, it may be remarked that the rule (24) $\Rightarrow$ (25) can still be generalized. Let the functions $h(x, m)$ and $g(x, j)$ be related by

$$h(x, m) = \sum_{j=0}^{m} t(m, j) g(x, j), \tag{33}$$

where the $t(m, j)$ are complex numbers. Define

$$\sum_{k=0}^{n} F(n, k) g(k, j) = \phi(n, j). \tag{34}$$

Then we have

$$\sum_{k=0}^{n} F(n, k) h(k, m) = \sum_{j=0}^{m} \phi(n, j) t(m, j). \tag{35}$$

This extended rule (34) $\Rightarrow$ (35) may even be used to obtain some interesting formulas involving Comtet's generalized Stirling numbers whose definitions may be found in [9]. However, we will omit the details here.

ACKNOWLEDGMENT

The author wishes to thank the referee for supplying useful comments with a remark.

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AMS numbers: 05A19, 05A10, 05A15

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