ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-481 ***Proposed by Richard André-Jeannin, Longwy, France***

Let $\phi(x)$ be the function defined by

$$\phi(x)=\sum_{n\geq 0}\frac{x^n}{F_{r^n}},$$

where $r\geq 2$ is a natural integer. Show that $\phi(x)$ is an irrational number, if x is a nonzero rational number.

H-482 ***Proposed by Larry Taylor, Rego Park, NY***

Let j, k, m, and n be integers. Let $A_n(m)=B_n(m-1)+4A_n(m-1)$ and $B_n(m)=4B_n(m-1)+5A_n(m-1)$ with initial values $A_n(0)=F_n$, $B_n(0)=L_n$.

(A) Generalize the numbers (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2) to form an eleven-term arithmetic progression of integral multiples of $A_{n+k}(m+j)$ and / or $B_{n+k}(m+j)$ with common difference $A_n(m)$.

(B) Generalize the numbers (3, 3, 3, 3, 3, 3, 3, 3, 3, 3) to form a ten-term arithmetic progression of integral multiples of $A_{n+k}(m+j)$ and / or $B_{n+k}(m+j)$ with common difference $A_n(m)$.

(C) Generalize the numbers (1, 1, 1, 1, 1, 1, 1, 1) to form an eight-term arithmetic progression of integral multiples of $A_{n+k}(m+j)$ and / or $B_{n+k}(m+j)$ with common difference $A_n(m)$.

Hint: $A_n(1)=-11(-1)^n A_{-n}(-1)$.

Reference: L. Taylor. Problem H-422. *The Fibonacci Quarterly* **28.3** (1990):285-87.

SOLUTIONS

A ... Periodic

H-464 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 30, no. 1, February 1992)***

Show that $\sum_{k=1}^{[n/2]}\binom{n}{k}A_{n-2k}=F_n$, where $A_j=(-1)^{[(j+2)/5]}-\left((-1)^{[j/5]}+(-1)^{[(j+4)/5]}\right)/2$. [ ] denotes the greatest integer function.

Solution by C. Georghiou, University of Patras, Patras, Greece

First, note that A_j is periodic with period 10 and with $A_0 = A_5 = 0$, $A_1 = A_2 = A_8 = A_9 = 1$, and $A_3 = A_4 = A_6 = A_7 = -1$. Its (ordinary) generating function is

$$\begin{aligned} g(z) &= (z+z^2-z^3-z^4-z^6-z^7+z^8+z^9)/(1-z^{10}) \\ &= (z+z^2-z^3-z^4)/(1+z^5) = z(1-z)(1+z)/(1-z+z^2-z^3+z^4) \\ &= \frac{z^{-1}-z}{z^2-z+1-z^{-1}+z^2}. \end{aligned}$$

Second, let

$$f(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/2]}\binom{n}{k}A_{n-2k}\right)x^n = \sum_{n,k=0}^{\infty}\binom{n+2k}{k}A_n x^{n+2k} = \sum_{n=0}^{\infty}A_n x^n\ {}_2F_1\left[\begin{matrix} n/2+1/2, n/2+1 \\ n+1 \end{matrix}; 4x^2\right],$$

where ${}_2F_1[\quad]$ is the Gauss hypergeometric series (see solution of H-444). But

$${}_2F_1\left[\begin{matrix} a,\ a+1/2 \\ 2a \end{matrix}; z\right] = 2^{2a-1}(1-z)^{-1/2}\left[1+(1-z)^{1/2}\right]^{1-2a},$$

(see M. Abramowitz & I. A. Stegun, *Handbook of Mathematical Functions* [New York: Dover, 1965] Entry 15.1.14, p. 556), and therefore, by setting $\partial = (1-4x^2)^{1/2}$ we obtain

$$f(x) = \frac{1}{\partial}\sum_{n=0}^{\infty}A_n\left(\frac{2x}{1+\partial}\right)^n = \frac{1}{\partial}g\left(\frac{2x}{1+\partial}\right).$$

Now

$$\frac{1+\partial}{2x} - \frac{2x}{1+\partial} = \partial / x \qquad \frac{1+\partial}{2x} + \frac{2x}{1+\partial} = 1/x$$

and

$$\left(\frac{1+\partial}{2x}\right)^2 + \left(\frac{2x}{1+\partial}\right)^2 = 1/x^2 - 2.$$

Therefore

$$f(x) = \frac{x}{1-x-x^2}$$

which is the generating function of F_n, and the assertion follows readily. Note that the problem is the same as H-444.

Also solved by P. Bruckman and the proposer.

B Good

H-465 ***Proposed by Richard André-Jeannin, Tunisia***
(Vol. 30, no. 1, February 1992)

Let p be a prime number, and let $r_1, r_2, \ldots, r_s$ be natural integers such that $s \ge 2, r_1 < p$, and $\sum_{k=1}^{k=s} kr_k = p$. Show that the number

$$B_{r_1, r_2, \ldots, r_s} = \frac{1}{r_1 + r_2 + \cdots + r_s} \frac{(r_1 + r_2 + \cdots + r_s)!}{r_1! r_2! \cdots r_s!}$$

is an integer.

Solution by Paul S. Bruckman, Edmonds, WA

Let $B_s \equiv B_{r_1, r_2, \ldots, r_s}$ for brevity. Let N denote the set of positive integers. We may express B_s as follows:

$$B_s = \frac{(r_1 + r_2 + \cdots + r_s - 1)!}{r_1! r_2! \cdots r_s!}. \tag{1}$$

From the condition $\sum_{k=1}^{s} kr_k = p$, with $1 \le r_k$, $k = 1, 2, \ldots, s$, it follows that $r_k < p$. Then, we see from (1) that $B_s = A / B$, say, where $\gcd(B, p) = 1$.

Also, there are s distinct ways to express B_s, as follows:

$$B_s = U_k / r_k, \quad k = 1, 2, \ldots, s, \tag{2}$$

where U_k is the multinomial coefficient defined as follows:

$$U_k = \frac{(r_1 + r_2 + \cdots + r_s - 1)!}{r_1! r_2! \cdots (r_k - 1)! \cdots r_s!}. \tag{3}$$

As we know, the U_k's are positive integers. Therefore, $r_k B_s \in N$. Therefore, $B_s \sum_{k=1}^{s} kr_k = pB_s \in N$. This implies that either $r_k B_s \in N$, or else $B_s = A / p$ for some integer A; however, as we have seen, this latter contingency is impossible, so we are done.

Also solved (partially) by the proposer.

A Unique Answer

<u>H-466</u> ***Proposed by Paul S. Bruckman, Edmonds, WA***
(Vol. 30, no. 2, May 1992)

Let p be a prime of the form $ax^2 + by^2$, where a and b are relatively prime natural numbers neither of which is divisible by p; x and y are integers. Prove that x and y are uniquely determined, except for trivial variations of sign.

Solution by Don Redmond, Southern Illinois University, Carbondale, IL

Suppose that there are two representations, say, $p = ax^2 + by^2$ and $p = ar^2 + bs^2$, where we may assume that x, y, r, and s are natural numbers. Then $(x, y) = (r, s) = 1$. If we eliminate b between the two representations, we have $p(y^2 - s^2) = a(r^2 y^2 - s^2 x^2)$.

Since $p \nmid a$, we see that $p | (r^2 y^2 - s^2 x^2)$, and so, for some choice of sign, we have

$$ry \equiv \pm sx \pmod{p}. \tag{1}$$

Also, the two representations give

$$p^2 = (ax^2 + by^2)(ar^2 + bs^2) = (axr \pm bys)^2 + ab(ry \mp sx)^2. \tag{2}$$

If $ry = sx$, then $(x, y) = 1 = (r, s)$ implies that $r = x$ and $s = y$.

If $ry \neq sx$, then (1) and (2) imply that $|ry \pm sx| = p, a = b = 1$, and $axr \pm bys = 0$. This implies, since $x^2 + y^2 = r^2 + s^2 = p$, that $x = s$ and $y = r$.

Thus p has essentially only one representation. □

Also solved by R. Isreal and the proposer.

Many Congruences

H-467 ***Proposed by Larry Taylor, Rego Park, NY***
(Vol. 30, no. 2, May 1992)

Let (a_n, b_n, c_n) be a primitive Pythagorean triple for n = 1, 2, 3, 4, where a_n, b_n, c_n are positive integers and b_n is even. Let $p \equiv 1 \pmod 8$ be prime; $r^2 + s^2 \equiv t^2 \pmod p$, where the Legendre symbol $\left(\frac{(t+r)/2}{p}\right) = 1$.

Solve the following twelve simultaneous congruences:

$$\begin{aligned}(a_1, b_1, c_1) &\equiv (r, s, t),\\ (a_2, b_2, c_2) &\equiv (r, s, -t),\\ (a_3, b_3, c_3) &\equiv (s, r, t),\\ (a_4, b_4, c_4) &\equiv (s, r, -t) \pmod p.\end{aligned}$$

For example, if $(r, s, t) \equiv (3, 4, 5) \pmod{17}$,

$$\begin{aligned}(a_1, b_1, c_1) &= (3, 4, 5),\\ (a_2, b_2, c_2) &= (105, 208, 233),\\ (a_3, b_3, c_3) &= (667, 156, 685),\\ (a_4, b_4, c_4) &= (21, 20, 29).\end{aligned}$$

Solution by Paul S. Bruckman, Edmonds, WA

All congruences are assumed to be (mod p), unless otherwise specified. Some definitions and notational remarks are in order. A pair of integers (u, v) is said to be a *generator* of the primitive Pythagorean triple (p.P.t.) (a, b, c) if the following conditions hold:

$$u > v > 0;\; u \not\equiv v \pmod 2;\; \gcd(u, v) = 1. \tag{1}$$

In that event, we have

$$a = u^2 - v^2;\; b = 2uv;\; c = u^2 + v^2. \tag{2}$$

We also write $(u, v) \in G(a, b, c)$, meaning that (u, v) satisfies (1), and (2) holds.

The hypothesis implies that r and t have the same parity, since $\left(\frac{\frac{1}{2}(t+r)}{p}\right) = 1$ is a stronger statement than $\left(\frac{2^{-1}(t+r)}{p}\right) = 1$; also, it is implied that s is even. Since $\left(\frac{1}{2}s\right)^2 \equiv \left[\frac{1}{2}(t+r)\right]\left[\frac{1}{2}(t-r)\right]$, it follows that $\left(\frac{\frac{1}{2}(t-r)}{p}\right) = 1$. Therefore, there exist integers u' and v' such that

$$(u')^2 \equiv \tfrac{1}{2}(t+r),\quad (v')^2 \equiv \tfrac{1}{2}(t-r). \tag{3}$$

By adding or subtracting the congruences in (3), we obtain

$$t \equiv (u')^2 + (v')^2,\quad r \equiv (u')^2 - (v')^2. \tag{4}$$

Also, $4(u'v')^2 \equiv t^2 - r^2 \equiv s^2$; thus, by an appropriate choice of signs for u' and / or v', we have

$$s \equiv 2u'v'. \tag{5}$$

There is nothing in the hypotheses to suggest that (r, s, t) is a p.P.t., even though $(r, s, t) = (3, 4, 5)$ in the example, which is indeed a p.P.t.; we could just as well have been given $(r, s, t) = (37, -30, 73)$, which also satisfies the hypotheses for $p = 17$, yet $37^2 + 30^2 \neq 73^2$. Nor is it likely that our initial choice of u' and v' satisfying (3) and (5) satisfy (1). However, we see that by adding suitable multiples of p to u' and / or v', we do obtain a new pair (u_1, v_1) that satisfies (1). It is then true that $(u_1, v_1) \in G(a_1, b_1, c_1)$, where $(a_1, b_1, c_1) \equiv (r, s, t)$. To use the data of the example, we may take $(u_1, v_1) = (2, 1)$ as the solution of (3) and (5), with $p = 17$, $(r, s, t) = (3, 4, 5)$, also satisfying (1), since $(2, 1) \in G(3, 4, 5)$.

Next, we observe that since $p \equiv 1 \pmod 8$, there exist solutions i and j of the following congruences:

$$i^2 \equiv -1, \quad j^2 \equiv 2^{-1}. \tag{6}$$

In fact, there are *two* solutions for each congruence in (6). We will need to choose the signs of i and j such that appropriate generators (u_n, v_n) may be found for (a_n, b_n, c_n), $n = 2, 3, 4$. Thus, for $n = 2$, and for an appropriate solution i of (6), we claim that (u_2, v_2) is found from the following:

$$u_2 \equiv iv_1, \quad v_2 \equiv -iu_1. \tag{7}$$

Proof: Given (7), then $u_2^2 - v_2^2 \equiv i^2(v_1^2 - u_1^2) \equiv u_1^2 - v_1^2 \equiv r$; $2u_2v_2 \equiv -2i^2u_1v_1 \equiv 2u_1v_1 \equiv s$; and $u_2^2 + v_2^2 \equiv i^2(u_1^2 + v_1^2) \equiv -u_1^2 - v_1^2 \equiv -t$. Also, we determine u_2 and v_2 that satisfy (1). It then follows that $(u_2, v_2) \in G(a_2, b_2, c_2)$, with $(a_2, b_2, c_2) \equiv (r, s, -t)$. In this example, we take $i \equiv -4$, $u_2 \equiv -4 \cdot 1$, $v_2 \equiv 4 \cdot 2$. We find that we may take $(u_2, v_2) = (13, 8)$, and that $(13, 8) \in G(105, 208, 233)$; also, $(105, 208, 233) \equiv (3, 4, -5)$.

Next, we claim that, by an appropriate choice of j, we have:

$$u_3 \equiv j(u_1 + v_1), \quad v_3 \equiv j(u_1 - v_1). \tag{8}$$

Proof: $u_3^2 - v_3^2 \equiv j^2 \cdot 4u_1v_1 \equiv 2u_1v_1 \equiv s$; $2u_3v_3 \equiv 2j^2(u_1^2 - v_1^2) \equiv u_1^2 - v_1^2 \equiv r$; and $u_3^2 + v_3^2 \equiv 2j^2(u_1^2 + v_1^2) \equiv u_1^2 + v_1^2 \equiv t$. In the example, $j \equiv 3$; then, $u_3 \equiv 3 \cdot 3, v_3 \equiv 3 \cdot 1$. We may take $(u_3, v_3) = (26, 3)$, and we find that this pair generates $(a_3, b_3, c_3) = (667, 156, 685) \equiv (4, 3, 5)$.

Finally, we claim that, for appropriate i and j, we have

$$u_4 \equiv ij(u_1 - v_1), \quad v_4 \equiv -ij(u_1 + v_1); \tag{9}$$

equivalently,

$$u_4 \equiv -iv_3, \quad v_4 \equiv iu_3. \tag{10}$$

Proof: $u_4^2 - v_4^2 \equiv i^2(v_3^2 - u_3^2) \equiv u_3^2 - v_3^2 \equiv s$; $2u_4v_4 \equiv -2i^2u_3v_3 \equiv 2u_3v_3 \equiv r$; and $u_4^2 + v_4^2 \equiv -i^2(v_3^2 + u_3^2) \equiv -u_3^2 - v_3^2 \equiv -t$. In this example, take $i \equiv 4$. Then $u_4 \equiv -4 \cdot 3 \equiv 5$ and $v_4 \equiv 4 \cdot 26 \equiv 2$. We find that $(5, 2) \in G(21, 20, 29)$, where $(21, 20, 29) \equiv (4, 3, -5)$.

To summarize, $(u_n, v_n) \in G(a_n, b_n, c_n)$, $n = 1, 2, 3, 4$, where

$$u_1^2 \equiv 2^{-1}(t+r),\ v_1^2 \equiv 2^{-1}(t-r);\ u_2 \equiv iv_1,\ v_2 \equiv -iu_1;$$
$$u_3 \equiv j(u_1+v_1),\ v_3 \equiv j(u_1-v_1);\ u_4 \equiv -iv_3,\ v_4 \equiv iu_3; \tag{11}$$

(u_1, v_1) and the values of i and j are obtained as appropriately chosen solutions of (3), (5), and (6), so as to satisfy (1) for each (u_n, v_n).

Also solved by the proposer.

A Very Odd Problem

H-468 ***Proposed by Lawrence Somer, Washington, D.C. (Vol. 30, no. 2, May 1992)***

Let $\{v_n\}_{0 \le n < \infty}$ be a Lucas sequence of the second kind satisfying the recursion relation $v_{n+2} = av_{n+1} + bv_n$, where a and b are positive odd integers and $v_0 = 2, v_1 = a$. Show that v_{2n} has an odd prime divisor $p \equiv 3 \pmod 4$ for $n \ge 1$.

Solution by Russell Jay Hendel, Patchogue, NY

If a is odd, then $a^2 \equiv 1 \pmod 4$ and $2a \equiv 2 \pmod 4$. It follows that the congruence classes modulo 4 of the sequence $v_0, v_1, v_2, \ldots,$ are $2, a, 3, a(3+b), 3, 3ab, 2, a, \ldots$. Since this sequence has period 6, $v_{6n\pm2} \equiv 3 \pmod 4$, implying that at least one of the prime factors of $v_{6n\pm2}$ is congruent to 3 modulo 4.

v_{2n} is either of the form v_{6n} or $v_{6n\pm2}$. Therefore, we have to deal with the case v_{2n}. First we note that $v_n | v_{nk}$ for any odd integer k. This follows because the Binet form of v_n is $\gamma^n + \delta^n$ with $\gamma = (a + \sqrt{\{a^2+4b\}})/2, \gamma + \delta = a. \gamma\delta = b$. Therefore, if k is an odd integer, the formula $x^k + y^k = (x+y)\{x^{k-1} + y^{k-1} - xy(x^{k-2} + y^{k-2}) + (xy)^2(x^{k-3} + y^{k-3}) \cdots \pm (xy)^{(k-1)/2}\}$ implies, with $x = \gamma^n$, $y = \delta^n$, that $v_n | v_{nk}$.

Proceeding as in [1], for each integer n, $6n = 2^m(6n'+3)$, for some integers m and n'. Since $2^m \equiv \pm 2 \pmod 6$, there is a prime $p \equiv 3 \pmod 4$ such that p divides v_{2^m}. Since $6n/2^m$ is odd, p also divides v_{2n} and the proof is complete.

Reference:

1. Sahib Singh. "Thoro's Conjecture and Allied Divisibility Property of Lucas Numbers." *The Fibonacci Quarterly* **18.2** (1980):135.

Also solved by P. Bruckman, R. André-Jeannin, and the proposer.