ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to 72717.3515@compuserve.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-748 ***Proposed by Herta T. Freitag, Roanoke, VA***

Let $u_k = F_{kn}/F_n$ for some fixed positive integer n. Find a recurrence satisfied by the sequence (u_k).

B-749 ***Proposed by Richard André-Jeannin, Longwy, France***

For n a positive integer, define the polynomial $P_n(x)$ by $P_n(x) = x^{n+2} - x^{n+1} - F_n x - F_{n-1}$. Find the quotient and remainder when $P_n(x)$ is divided by $x^2 - x - 1$.

B-750 ***Proposed by Seung-Jin Bang, Albany, CA***

Find a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(F_n, L_n) = (F_{n+1}, L_{n+1})$.

B-751 ***Proposed by Jayantibhai M. Patel, Bhavan's R. A. College of Science, Gujarat State, India***

Prove that $6L_{n+3}L_{3n+4} + 7$ and $6L_nL_{3n+5} - 7$ are divisible by 25.

B-752 ***Proposed by Richard André-Jeannin, Longwy, France***

Consider the sequences (U_n) and (V_n) defined by the recurrences $U_n = PU_{n-1} - QU_{n-2}$, $n \ge 2$, with $U_0 = 0, U_1 = 1$, and $V_n = PV_{n-1} - QV_{n-2}$, $n \ge 2$, with $V_0 = 2, V_1 = P$, where P and Q are real numbers with $P > 0$ and $\Delta = P^2 - 4Q > 0$. Show that for $n \ge 0, U_{n+1} \ge (P/2)U_n$ and $V_{n+1} \ge (P/2)V_n$.

B-753 ***Proposed by Jayantibhai M. Patel, Bhavan's R. A. College of Science, Gujarat State, India***

Prove that, for all positive integers n,

$$\begin{vmatrix} F_{n-1}^3 & F_n^3 & F_{n+1}^3 & F_{n+2}^3 \\ F_n^3 & F_{n+1}^3 & F_{n+2}^3 & F_{n+3}^3 \\ F_{n+1}^3 & F_{n+2}^3 & F_{n+3}^3 & F_{n+4}^3 \\ F_{n+2}^3 & F_{n+3}^3 & F_{n+4}^3 & F_{n+5}^3 \end{vmatrix} = 36.$$

SOLUTIONS

Convolution Solution

B-720 ***Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy (Vol. 30, no. 3, August 1992)***

Find a closed form expression for $S_n = \sum_{h+k=2n} F_h F_k$, where the sum is taken over all pairs of positive integers (h, k) such that $h+k=2n$ and $h \le k$.

Solution by Russell Euler, Northwest Missouri State University, Maryville, MO

Using the Binet formula, we have

$$S_n = \sum_{h=1}^{n} F_h F_{2n-h} = \frac{1}{5}\sum_{h=1}^{n}\left[\alpha^{2n} + \beta^{2n} - \beta^{2n}\left(\frac{\alpha}{\beta}\right)^h - \alpha^{2n}\left(\frac{\beta}{\alpha}\right)^h\right]$$

$$= \frac{1}{5}\left[nL_{2n} - \beta^{2n}\frac{(\alpha/\beta) - (\alpha/\beta)^{n+1}}{1-\alpha/\beta} - \alpha^{2n}\frac{(\beta/\alpha) - (\beta/\alpha)^{n+1}}{1-\beta/\alpha}\right].$$

The sum was evaluated by the standard formula for the sum of a geometric progression:

$$\sum_{h=1}^{n} r^h = \frac{r - r^{n+1}}{1-r}.$$

Upon simplifying, we find that

$$S_n = \frac{1}{5}\left[nL_{2n} + F_{2n-1} - (-1)^n\right].$$

Most sums of this form can be found by the same method. Other, equivalent formulas found by solvers were: $(nL_{2n} + F_n L_{n-1})/5$, $nF_n^2 + [F_{2n-1} + (-1)^n(2n-1)]/5$, $[(n+1)L_{2n} - L_n F_{n+1}]/5$, $(n+1)F_n^2 - [F_{2n+1} - (2n+1)(-1)^n]/5$, and $[(5n+1)L_{2n} + L_{2n-2} - 5(-1)^n]/25$.

Seiffert mentions the related convolution ([1], p. 118):

$$\sum_{j=1}^{2n-1} F_j F_{2n-j} = \frac{1}{5}\left[(2n-1)F_{2n+1} + (2n+1)F_{2n-1}\right].$$

Reference:

1. V. E. Hoggatt, Jr., & Marjorie Bicknell-Johnson. "Fibonacci Convolution Sequences." *The Fibonacci Quarterly* **15.2** (1977):117-22.

Also solved by Paul S. Bruckman, Russell Euler, Graham Lord, Dorka Ol. Popova, Bob Prielipp, H.-J. Seiffert, Tony Shannon, Sahib Singh, and the proposer.

Brittany Climbs Some Stairs

B-721 ***Proposed by Russell Jay Hendel, Dowling College, Oakdale, NY (Vol. 30, no. 3, August 1992)***

Brittany is going to ascend an m step staircase. At any time she is just as likely to stride up one step as two steps. For a positive integer k, find the probability that she ascends the whole staircase in k strides.

Editor's Comment: *Only one correct solution was received. We therefore begin by analyzing where most solvers went wrong.*

Those "solvers" fell into two camps: Camp A believes the answer is $\binom{k}{m-k}/2^k$; Camp B believes the answer is $\binom{k}{m-k}/F_{m+1}$. Both camps agree that the number of distinct ascents with k strides is $\binom{k}{m-k}$ and that the total number of different ways of climbing the stairs is F_{m+1} (see [2], p. 10).

Let us look at a staircase with 3 steps (the case $m = 3$). Camps A and B would have us believe the probabilities as shown in the corresponding tables below. In these tables, $p(k)$ denotes the probability that Brittany ascends in k strides.

k	1	2	3
$p(k)$	0	$\frac{2}{4}$	$\frac{1}{8}$

Camp A

k	1	2	3
$p(k)$	0	$\frac{2}{3}$	$\frac{1}{3}$

Camp B

Camp A cannot be correct because their probabilities do not add up to 1.

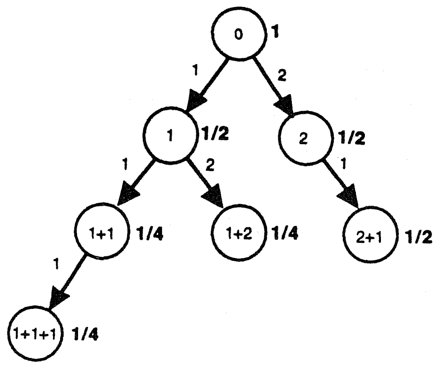
Camp B notes that there are 3 types of ascents, $2+1, 1+2$, and $1+1+1$. They assume each method of ascent is equally likely. Since there are 2 ascents of length 2 and 1 ascent of length 3, this determines the probabilities shown in their table above. The probabilities add up to 1. However, Camp B cannot be correct because they believe ascents that begin with a stride of 1 step ($1+2$ and $1+1+1$) occur twice as often as ascents that begin with a stride of 2 steps ($2+1$). Yet we know that on Brittany's first stride, she is just as likely to stride up 1 step as 2 steps.

To try to settle this discrepancy, I decided to watch Brittany during the past year and record data about her ascents. Fortunately, she climbs the Tower of London frequently, and I was able to record data on 4000 ascents of 3-step staircases. It turns out that 3015 ascents were of length 2 and 985 were of length 3. This suggests that the correct probabilities are those given in the table below:

k	1	2	3
$p(k)$	0	$\frac{3}{4}$	$\frac{1}{4}$

Observed Probabilities

These probabilities can be confirmed by the following transition tree. Each node represents a state during the ascent. The edges show whether Brittany makes a stride of 1 or 2 steps from each state. The probability of reaching each state is given to the right of that state. We assume that all paths out of a state are equally likely, since at any time, Brittany is just as likely to stride up 1 step as 2 steps. Of course, when Brittany is one step away from her goal, she is forced to make a final stride of 1 step.



One can see from this transition tree that the probability of a 3-stride ascent is 1/4. Next, we move on to the general solution.

Solution by Peter Griffin, California State University, Sacramento, CA

There are two mutually exclusive ways to ascend m steps in k strides.

If the final stride was a double-step, then Brittany made $m-k-1$ double-steps in her first $k-1$ strides. The number of ways this could happen is $\binom{k-1}{m-k-1}$. Each of the k strides in the complete ascent occurred with probability 1/2.

If the final stride was a single step, then the last stride was forced and thus was taken with probability 1. In her first $k-1$ strides, Brittany must have made $m-k$ double-steps (each with probability 1/2). The number of ways this could happen is $\binom{k-1}{m-k}$.

Thus, the probability of ascending the whole staircase in k strides is

$$\frac{\binom{k-1}{m-k-1}}{2^k}+\frac{\binom{k-1}{m-k}}{2^{k-1}}=\frac{3k-m}{k}\binom{k}{m-k}\Big/2^k.$$

The proposer indicated that his proposal generalizes Problem 10 on page 407 of [1] and that this problem is a natural example of a discrete probability space that can be represented by a tree whose paths are not all the same length.

References:

1. Billstein, Libeskind, & Lott. *Mathematics for Elementary School Teachers: A Problem Solving Approach.* 4th ed. Redwood City, CA: Benjamin/Cummings, 1990.
2. S. Vajda. *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Four incorrect solutions were received.

Fibonacci Integrand

B-722 *Proposed by H.-J. Seiffert, Berlin Germany (Vol. 30, no. 3, August 1992)*

Define the Fibonacci polynomials by $F_0(x) = 0$, $F_1(x) = 1$, and $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, for $n \geq 2$. Show that for all nonnegative integers n,

$$\int_0^\infty \frac{dx}{(x^2+1)F_{2n+1}(2x)} = \frac{\pi}{4n+2}.$$

Solution by Hans Kappus, Rodersdorf, Switzerland

The Binet formula for the Fibonacci polynomials ([2], p. 99) is

$$F_n(x) = \frac{1}{\sqrt{x^2+4}}\left[\left(\frac{x+\sqrt{x^2+4}}{2}\right)^n - \left(\frac{x-\sqrt{x^2+4}}{2}\right)^n\right].$$

Thus, the integral turns out to be

$$I_n = \int_0^\infty \frac{2dx}{\sqrt{x^2+1}[(x+\sqrt{x^2+1})^{2n+1} - (x-\sqrt{x^2+1})^{2n+1}]}.$$

The substitution $x = \sinh t$ gives

$$I_n = \int_0^\infty \frac{dt}{\cosh[(2n+1)t]}.$$

Finally, the substitution $t = \theta/(2n+1)$ gives

$$I_n = \frac{1}{2n+1}\int_0^\infty \operatorname{sech}\theta\, d\theta = \frac{1}{2n+1}[\arctan(\sinh\theta)]_0^\infty = \frac{\pi}{4n+2}.$$

We have used the following well-known results about hyperbolic functions (see §4.5 of [1]):

$$\frac{d\sinh z}{dz} = \cosh z,\ \cosh^2 z - \sinh^2 z = 1,\ (\cosh z + \sinh z)^n = \cosh nz + \sinh nz,$$

and

$$\int \operatorname{sech} z\, dz = \arctan(\sinh z).$$

References:

1. Milton Abramowitz & Irene A. Stegun. *Handbook of Mathematical Functions.* Washington, D.C.: National Bureau of Standards, 1964.
2. P. Filipponi & A. F. Horadam. "Derivative Sequences of Fibonacci and Lucas Polynomials." In *Applications of Fibonacci Numbers.* Vol. 4, pp. 99-108. Dordrecht: Kluwer, 1991.

Also solved by Seung-Jin Bang, Paul S. Bruckman, Piero Filipponi, Igor Ol. Popov, and the proposer.

The Great Divide

B-723 ***Proposed by Bruce Dearden & Jerry Metzger, U. of North Dakota, Grand Forks, ND (Vol. 30, no. 3, August 1992)***

(a) Show that, for $n \equiv 2 \pmod 4$, $F_{n+1}(F_n^2 + F_n - 1)$ divides $F_n^n(F_n^2 + F_{n+1}) - 1$.

(b) What is the analog of (a) for $n \equiv 0 \pmod 4$?

Solution by H.-J. Seiffert, Berlin, Germany

It is easily verified that, for all positive integers k,

$$\sum_{j=0}^{2k-1} F_{j+1}x^j = \frac{F_{2k}x^{2k+1} + F_{2k+1}x^{2k} - 1}{x^2 + x - 1}, \tag{1}$$

$$\sum_{r=0}^{k-1} (-1)^r F_{2r+1} = \frac{2 - (-1)^k L_{2k}}{5}, \tag{2}$$

$$\sum_{r=0}^{k-1} (-1)^r F_{2r+2} = \frac{1 - (-1)^k L_{2k+1}}{5}, \tag{3}$$

$$2 + F_{4k+2} + L_{4k+2} + F_{4k+2}L_{4k+3} = 5F_{4k+3}F_{2k+1}^2, \tag{4}$$

and

$$F_n^2 = F_{n-1}F_{n+1} - (-1)^n. \tag{5}$$

(a) If n is a positive integer with $n \equiv 2 \pmod 4$, then there exists a nonnegative integer k, such that $n = 4k + 2$. Now in formula (1), replace k by $2k + 1$ and set $x = F_{4k+2}$ to obtain

$$Q = \frac{F_{4k+2}^{4k+2}(F_{4k+2}^2 + F_{4k+3}) - 1}{F_{4k+2}^2 + F_{4k+2} - 1} = \sum_{j=0}^{4k+1} F_{j+1}F_{4k+2}^j.$$

Note that the left side of this equation, Q, is an integer, showing that $F_n^n(F_n^2 + F_{n+1}) - 1$ is divisible by $F_n^2 + F_n - 1$. It remains to show that Q is divisible by F_{n+1}.

Using result (5) with $n = 4k + 2$, we find

$$Q = \sum_{r=0}^{2k} F_{2r+1}F_{4k+2}^{2r} + \sum_{r=0}^{2k} F_{2r+2}F_{4k+2}^{2r+1} \equiv \sum_{r=0}^{2k} (-1)^r F_{2r+1} + F_{4k+2}\sum_{r=0}^{2k} (-1)^r F_{2r+2} \pmod{F_{4k+3}}.$$

Applying results (2) and (3) followed by (4) gives

$$Q \equiv \frac{2 + F_{4k+2} + L_{4k+2} + F_{4k+2}L_{4k+3}}{5} \equiv F_{4k+3}F_{2k+1}^2 \equiv 0 \pmod{F_{4k+3}}.$$

Thus, Q is divisible by $F_{n+1} = F_{4k+3}$.

(b) By the same method, one can prove that, for $n \equiv 0 \pmod 4$,

$$F_{n+1}(F_n^2 - F_n - 1) \text{ divides } F_n^n(F_n^2 - F_{n+1}) + 1.$$

Also solved by Paul S. Bruckman and the proposers.

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