REAL FIBONACCI AND LUCAS NUMBERS WITH REAL SUBSCRIPTS

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1. INTRODUCTION

Several definitions of Fibonacci and Lucas numbers with real subscript are available in literature. In general, these definitions give complex quantities when the subscript is not an integer [1], [3], [8], [9].

In this paper we face, from a rather general point of view, the problem of defining numbers F_x and L_x which are *real* when subscript x is real. In this kind of definition, the minimum requirement is, obviously, that F_x and L_x and the usual Fibonacci numbers F_n and Lucas numbers L_n coincide when $x = n$ is an integer. Further, for all x, the fulfillment of some of the main properties possessed by F_n and L_n is desirable. Some of these definitions have already been given by other authors (e.g., [6], [10]).

Here, after a brief discussion on some general aspects of these definitions, we propose two distinct expressions for both F_x and L_x and study some of their properties. More precisely, in Section 2 we give an *exponential* representation for F_x and L_x, whereas in Section 3 we give a *polynomial* representation for these numbers. In spite of the fact that the numbers defined in the above said ways coincide only when x is an integer, they are denoted by the same symbol. Nevertheless, there is no danger of confusion since each definition applies only to the proper section.

We confine ourselves to consider only *nonnegative* values of the subscript, so that in all the statements involving numbers of the form F_{x-y} and L_{x-y} *it is understood that* $y \leq x$. The following notation is used throughout the paper:

$\lambda(x)$, the greatest integer not exceeding x,
$\mu(x)$, the smallest integer not less than x.

2. EXPONENTIAL REPRESENTATION OF F_x AND L_x

Keeping in mind the Binet forms for F_n and L_n leads, quite naturally to consideration of expressions of the following types:

$$F_x = [\alpha^x - f(x)\alpha^{-x}]/\sqrt{5} \tag{2.1}$$

and

$$L_x = \alpha^x + f(x)\alpha^{-x}, \tag{2.2}$$

where $\alpha = (1+\sqrt{5})/2$ is the positive root of the equation $z^2 - z - 1 = 0$, and $f(x)$ is a function of the real variable x such that

$$f(n) = (-1)^n \text{ for all integers } n. \tag{2.3}$$

It is plain that the numbers F_x and L_x defined by (2.1)-(2.3) and the usual Fibonacci numbers F_n and Lucas numbers L_n coincide whenever $x = n$ is an integer.

If we require that F_x and L_x enjoy some of the properties of F_n and L_n, we must require that $f(x)$ has some additional properties beyond that stated in (2.3).

Theorem 1: If, for all x,

$$f(x+1) = -f(x) \tag{2.4}$$

then the fundamental relations

$$F_{x+2} = F_{x+1} + F_x \tag{2.5}$$

and

$$L_{x+2} = L_{x+1} + L_x \tag{2.6}$$

are satisfied.

Proof: By (2.2) and (2.4), we can write

$$\begin{aligned} L_{x+1} + L_x &= \alpha^{x+1} + f(x+1)\alpha^{-x-1} + \alpha^x + f(x)\alpha^{-x} \\ &= \alpha^{x+1} + \alpha^x + f(x)(\alpha^{-x} - \alpha^{-x-1}). \end{aligned}$$

Since $\alpha^2 = \alpha + 1$ and $\alpha^{-2} = 1 - \alpha^{-1}$, we have $\alpha^{x+1} + \alpha^x = \alpha^{x+2}$ and $\alpha^{-x} - \alpha^{-x-1} = \alpha^{-x-2}$. Thus,

$$L_{x+1} + L_x = \alpha^{x+2} + f(x)\alpha^{-x-2} + f(x+2)\alpha^{-x-2} = L_{x+2}. \text{ Q.E.D.}$$

Theorem 2: If, for a particular x,

$$f^2(x) = f(2x), \tag{2.7}$$

then the identity

$$F_x L_x = F_{2x} \tag{2.8}$$

is satisfied.

Proof: By (2.1) and (2.2), after some simple manipulations, we get

$$F_x L_x = [\alpha^{2x} - f^2(x)\alpha^{-2x}]/\sqrt{5}. \text{ Q.E.D.}$$

Theorem 3: If the condition (2.4) is satisfied for all x, then the identity

$$L_x = F_{x-1} + F_{x+1} \tag{2.9}$$

holds.

The proof of Theorem 3 is analogous to that of Theorem 1 and is omitted for brevity.

Parker [10] used the function

$$f(x) = \cos(\pi x) \tag{2.10}$$

to obtain real Fibonacci and Lucas numbers with real subscripts. The function (2.10) satisfies (2.3) and (2.4) but does not satisfy (2.7). Other circular functions (or functions of circular functions) might be used as $f(x)$. For example, $f(x) = \cos^k(\pi x)$ and $f(x) = \cos^{1/k}(\pi x)$ (k an odd integer) satisfy the above properties as well. Further functions might be considered. For example, the function

$$f(x) = [a^2 - \sin^2(\pi x)]^{1/2} - a + \cos(\pi x) \quad (a \geq 1),$$

which describes the piston stroke as a function of the crank angle πx and the ratio a of the rod length to the crank radius, satisfies (2.3) but does not satisfy (2.4).

In my opinion, the simplest function $f(x)$ satisfying (2.3) and (2.4) is the function

$$f(x) = (-1)^{\lambda(x)} \tag{2.12}$$

which leads to the definitions

$$F_x = [\alpha^x - (-1)^{\lambda(x)} \alpha^{-x}] / \sqrt{5} \tag{2.13}$$

and

$$L_x = \alpha^x + (-1)^{\lambda(x)} \alpha^{-x}. \tag{2.14}$$

Observe that (2.12) can be viewed as a particular function of circular functions. In fact, this function and the special Fourier series

$$f(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\pi x]}{2k+1} \tag{2.12'}$$

coincide, except for the integral values of x.

As an illustration, the behavior of F_x vs x is shown in Figure 1 for $0 \leq x \leq 10$.

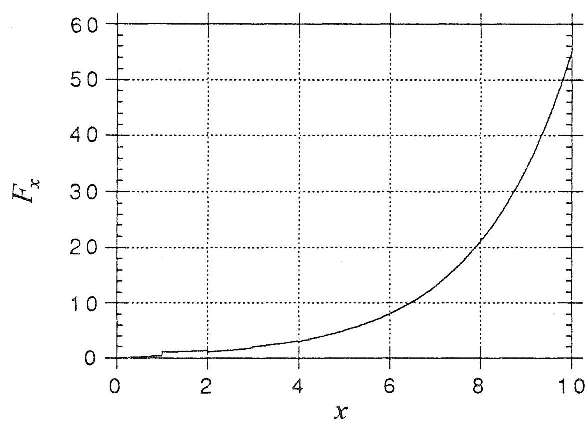

FIGURE 1. Behavior of F_x vs x for $0 \leq x \leq 10$

The discontinuities (observable for small values of x) connected with the integral values of x are obviously due to the greatest integer function inherent in the definition (2.13).

The numbers F_x and L_x defined by (2.13) and (2.14), respectively, enjoy several properties of the usual Fibonacci and Lucas numbers. For example, the following two propositions can be stated.

Proposition 1: $5F_x^2 = L_x^2 - 4(-1)^{\lambda(x)}$.

Proposition 2 (Simson formula analog): $F_{x-1}F_{x+1} - F_x^2 = (-1)^{\lambda(x)}$.

For the sake of brevity, we shall prove only Proposition 2.

Proof of Proposition 2: From (2.13), we can write

$$\begin{aligned} F_{x-1}F_{x+1}-F_x^2 &= [\alpha^{2x}-(-1)^{\lambda(x+1)}\alpha^{-2}-(-1)^{\lambda(x-1)}\alpha^2+(-1)^{\lambda(x-1)+\lambda(x+1)}\alpha^{-2x} \\ &\qquad -\alpha^{2x}-\alpha^{-2x}+2(-1)^{\lambda(x)}]/5 \\ &= [-(-1)^{\lambda(x)+1}\alpha^{-2}-(-1)^{\lambda(x)-1}\alpha^2+(-1)^{2\lambda(x)}\alpha^{-2x}-\alpha^{-2x}+2(-1)^{\lambda(x)}]/5 \\ &= [(-1)^{\lambda(x)}(\alpha^2+\alpha^{-2})+2(-1)^{\lambda(x)}]/5 \\ &= (-1)^{\lambda(x)}(L_2+2)/5=(-1)^{\lambda(x)}. \text{ Q.E.D.} \end{aligned}$$

Let us conclude this section by offering the sums of some finite series involving the numbers F_x and L_x. These are

$$\sum_{k=0}^{n} T_{x+k} = T_{n+x+2} - T_{x+1}, \tag{2.15}$$

where T stands for both F and L, and

$$\sum_{k=1}^{n} F_{k/n} = 1 + \frac{1}{\sqrt{5}(L_{1/n}-2)} - \frac{F_{(n-1)/n}+F_{1/n}}{L_{1/n}-2} \quad (n \geq 2), \tag{2.16}$$

$$\sum_{k=1}^{n} L_{k/n} = \frac{\sqrt{5}-L_{(n-1)/n}}{L_{1/n}-2} \quad (n \geq 2). \tag{2.17}$$

The proofs of (2.15)-(2.17) can be obtained from (2.13) and (2.14) with the aid of the geometric series formula. They are left to the interested reader.

3. POLYNOMIAL REPRESENTATION OF F_x AND L_x

Let us recall the well-known formula

$$F_n = \sum_{j=0}^{U} \binom{n-1-j}{j}, \tag{3.1}$$

where U is a suitable integral function of n, which gives the n^{th} Fibonacci number. It is also well known (see, e.g., [5, p. 48]) that the binomial coefficient defined as

$$\binom{a}{0} = 1, \ \binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!} \quad (k \geq 1 \text{ an integer}) \tag{3.2}$$

makes sense also if a is any real quantity.

In light of (3.2), some conditions must be imposed on the upper range indicator, U, for (3.1) to be efficient. In my opinion, the usual choice $U=\infty$ (see, e.g., [13, (54)]) is not correct. For example, for $n=5$ and $U=\infty$, we have the infinite series

$$F_5 = \binom{4}{0}+\binom{3}{1}+\binom{2}{2}+\binom{1}{3}+\binom{0}{4}+\binom{-1}{5}+\binom{-2}{6}+\binom{-3}{7}+\binom{-4}{8}+\cdots$$
$$= 1 + 3 + 1 + 0 + 0 - 1 + 7 - 36 + 165 - \cdots$$

the sum of which is clearly different from 5. It can be readily proved that formula (3.1) works correctly if the following inequalities are satisfied

$$\lambda[(n-1)/2] \le U \le n-1. \tag{3.3}$$

On the basis of (3.2) and (3.3), a polynomial representation of F_x can be obtained by simply replacing n by x in (3.1). Following the choice of Schroeder [12, p. 68] (i.e., $U = \lambda[(n-1)/2]$), we define the numbers F_x as

$$F_x = \sum_{j=0}^{\lambda[(x-1)/2]} \binom{x-1-j}{j} \tag{3.4}$$

Observe that, under the convention that a sum vanishes when the upper range indicator is smaller than the lower one and taking into account that $\lambda(-x) = -\mu(x)$, expression (3.4) allows us to obtain $F_0 = 0$.

Other choices of U are possible, within the interval (3.3). In a recent paper [1] André-Jeannin considered the numbers $G(x)$ ($x > 0$) obtained by replacing n by x and U by $m(x)$ in (3.1), $m(x)$ being the integer defined by $x/2-1 \le m(x) < x/2$. It is readily seen that $m(x) = \mu(x/2-1)$, and $m(x) = \lambda[(x-1)/2]$ when x is an integer. Moreover, we can see that F_x and $G(x)$ coincide for $2h-1 \le x \le 2h$ (h = 1, 2, ...), and both of them give the usual Fibonacci numbers F_n when $x = n$ is an integer.

As an illustration, we give the value of F_x for $0 \le x < 9$.

$$\begin{aligned}
&F_x = 0, \text{ for } 0 \le x < 1,\\
&F_x = 1, \text{ for } 1 \le x < 3,\\
&F_x = x-1, \text{ for } 3 \le x < 5,\\
&F_x = (x^2-5x+10)/2, \text{ for } 5 \le x < 7,\\
&F_x = (x^3-12x^2+59x-90)/6, \text{ for } 7 \le x < 9.
\end{aligned}$$

The behavior of F_x vs x for $0 \le x < 9$ is shown in Figure 2 below.

Replacing n by x in [4, (1.3)-(1.4)] leads to an analogous polynomial representation of L_x:

$$L_x = \sum_{j=0}^{\lambda(x/2)} \frac{x}{x-j}\binom{x-j}{j}. \tag{3.5}$$

Observe that, for x = 0, this definition gives the indeterminate form 0/0. So, $L_0 = 2$ cannot be defined by (3.5). As an illustration, we show the values of L_x for $0 < x < 8$.

$$\begin{aligned}
&L_x = 1, \text{ for } 0 < x < 2,\\
&L_x = x+1, \text{ for } 2 \le x < 4,\\
&L_x = (x^2-x+2)/2, \text{ for } 4 \le x < 6,\\
&L_x = (x^3-6x^2+17x+6)/6, \text{ for } 6 \le x < 8.
\end{aligned}$$

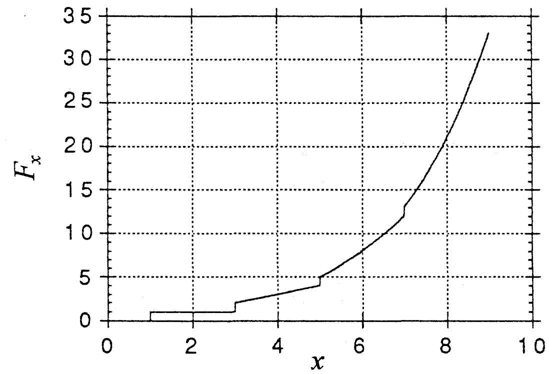

FIGURE 2. Behavior of F_x vs x for $0 \le x < 9$

Also the numbers F_x and L_x defined by (3.4) and (3.5), respectively, enjoy several properties of the usual Fibonacci and Lucas numbers. Sometimes these properties hold for all x, but, in most cases, their validity depends on the parity of $\lambda(x)$. We shall give an example for each case. The proof of the latter is omitted for brevity.

Proposition 3: $F_{x-1} + F_{x+1} = L_x$.

Proof: From (3.5), the binomial identity available in [11, p. 64], and (3.4), we can write

$$L_x = \sum_{j=0}^{\lambda(x/2)} \left[\binom{x-j}{j} + \binom{x-1-j}{j-1} \right] = F_{x+1} + \sum_{j=0}^{\lambda(x/2)} \binom{x-1-j}{j-1} = F_{x+1} + \sum_{j=-1}^{\lambda(x/2)-1} \binom{x-2-j}{j}.$$

By virtue of the assumption [5, p. 48],

$$\binom{x}{-k} = 0 \quad (k \ge 1 \text{ an integer}), \tag{3.6}$$

by using the equality

$$\lambda(x/2) - 1 = \lambda[(x-2)/2], \tag{3.7}$$

and definition (3.4), the previous expression becomes

$$L_x = F_{x+1} + \sum_{j=0}^{\lambda[(x-2)/2]} \binom{x-2-j}{j} = F_{x+1} + F_{x-1}. \quad \text{Q.E.D.}$$

Proposition 4:

$$F_x + F_{x+1} = \begin{cases} F_{x+2}, & \text{if } \lambda(x) \text{ is even,} \\ F_{x+2} - \binom{x - \lambda(x/2) - 1}{\lambda(x/2) + 1}, & \text{if } \lambda(x) \text{ is odd.} \end{cases}$$

Let us conclude this section by considering a special case [namely, $n = (2k+1)/2$] of the well-known identity $F_n L_n = F_{2n}$. The numerical evidence shows that

$$F_{(2k+1)/2} L_{(2k+1)/2} = F_{2k+1} - g(k). \tag{3.8}$$

The values of $g(k)$ for the first few values of k are shown below:

$$\begin{array}{ll} g(0)=1 & g(4)=2.9375 \\ g(1)=1 & g(5)=3.734375 \\ g(2)=1.5 & g(6)=6.4921875 \\ g(3)=1.75 & g(7)=8.57421875. \end{array}$$

I was able to find neither a closed form expression nor sufficiently narrow bounds for $g(k)$. Establishing an expression for this quantity is closely related to the problem of expressing F_x and L_x as functions of $F_{\lambda(x)}$ and $L_{\lambda(x)}$, respectively ($x = k + 1/2$ in the above case). This seems to be a challenging problem the solution of which would allow us to find many more identities involving the numbers F_x and L_x. Any contribution of the readers on this topic will be deeply appreciated.

4. CONCLUDING REMARKS

In this paper we have proposed an exponential representation and a polynomial representation for Fibonacci numbers F_x and Lucas numbers L_x that are real if x is real. Some of their properties have also been exhibited.

As for the polynomial representation, we point out that other sums, beyond (3.1) and [4, (1.3)-(1.4)] [see (3.5)], give the Fibonacci and Lucas numbers. These sums can be used to obtain further polynomial representations for F_x and L_x. For example, if we replace n by x in the expression for Fibonacci numbers available in [2], we have

$$F_x = \sum_{j=-\lambda(x/5)}^{\lambda[(x-1)/5]} (-1)^j \binom{x-1}{\lambda[(x-1-5j)/2]}. \tag{4.1}$$

Observe that (4.1) and (3.4) coincide for $0 \le x < 5$. Getting the polynomials in x given by (4.1) for higher values of x, requires a lot of tedious calculations. As an illustration, we give the value of F_x for $0 \le x < 8$.

$$\begin{aligned} &F_x = 0, \text{ for } 0 \le x < 1, \\ &F_x = 1, \text{ for } 1 \le x < 3, \\ &F_x = x-1, \text{ for } 3 \le x < 5, \\ &F_x = (-x^4+10x^3-23x^2+14x)/24, \text{ for } 5 \le x < 6, \\ &F_x = (-x^5+15x^4-85x^3+285x^2-454x+120)/120, \text{ for } 6 \le x < 7, \\ &F_x = (-x^5+15x^4-65x^3+105x^2-54x-120)/120, \text{ for } 7 \le x < 8. \end{aligned}$$

Plotting these values shows clearly that definition (4.1) is rather unsatisfactory if compared with definition (3.4). We reported definition (4.1) here for the sake of completeness and because it might be interesting *per se*.

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EDITOR ON LEAVE OF ABSENCE

The Editor has been asked to visit Yunnan Normal University in Kunming, China, for the Fall semester of 1993. This is an opportunity that the Editor and his wife feel cannot be turned down. They will be in China from August 1, 1993, until approximately January 10, 1994. The August and November issues of *The Fibonacci Quarterly* will be delivered to the printer early enough so that these two issues can be published while the Editor is out of the country. The Editor has also arranged for several individuals to send out articles to be refereed which have been submitted for publication in *The Fibonacci Quarterly* or submitted for presentation at the *Sixth International Conference on Fibonacci Numbers and Their Applications.* Things may be a little slower than normal, but every attempt will be made to insure that all goes as smoothly as possible while the Editor is on leave in China. PLEASE CONTINUE TO USE THE NORMAL ADDRESS FOR SUBMISSION OF PAPERS AND ALL OTHER CORRESPONDENCE.